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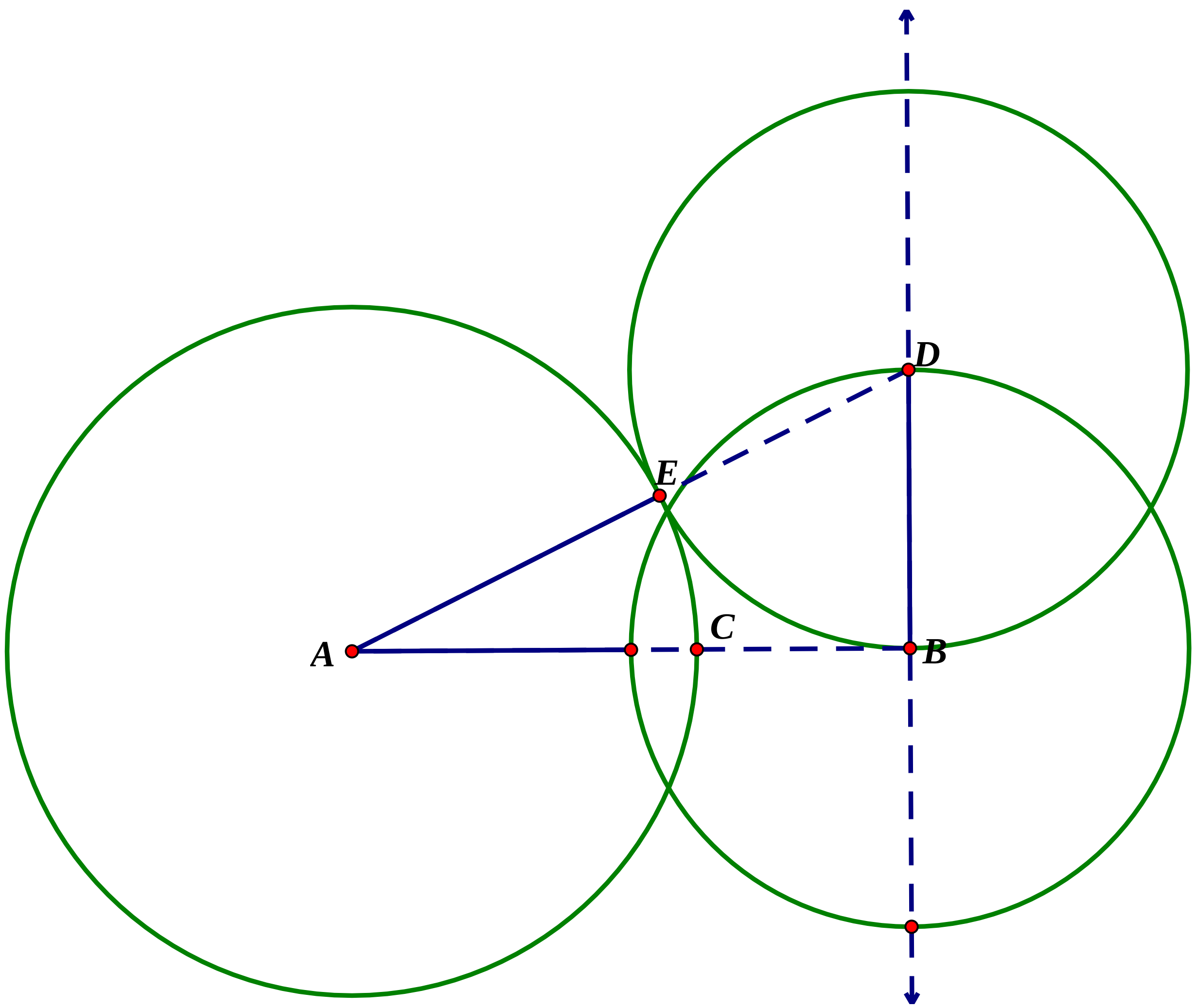
The Golden Ratio

The world we live in is extremely complex. Scientists, mathematicians, and historians have been searching for truths to explain these complexities since the beginning of time. Many people have spent their entire lives searching for patterns and relationships in nature. In the Divine Proportion we see a relationship in nature that creates balance and symmetry that has fascinated mathematicians and artists for centuries. In words, the Divine Proportion states that the whole is to the larger in exactly the same proportion as the larger is to the smaller. It can also be described by the Fibonocci Sequence in which the next term is found by adding the two previous terms. These relationships are interesting, but what does this tell us and why are they famous? These relationships suggest that there is a relationship between numbers and nature. These relationships among numbers appear in shapes, patterns, and dynamics in nature. This balance provides the foundations for things as simple as a work of art or as complicated to a strand of DNA. There are many aspects to the Divine Ratio and why the mysteries behind it have captured the eye of so many people before our time (Hemenway, 2005, p.11-27).

The Divine Proportion has been impressing researchers for centuries. This can explain the many names that all describe the same idea. The Divine Proportion is also called, Golden Mean, Golden Section, Golden Ratio, Golden Proportion, and Sacred Cut. These all describe the proportion that is mathematically defined as Φ (phi). Once again, this proportion is the relationship that two parts, one larger and one smaller, are to each as the whole is to the larger part. Historians do not know exactly when the discovery of the Golden Ratio was made. One can infer that the many names given to this relationship suggests that it was discovered multiple times by multiple groups of people. There is some evidence that the Golden Ration was used by the Egyptians to design the pyramids. It is also believed that the Greeks used it to construct Parthenon. Famous mathematicians such as Plato, Euclid, Fibonacci, and Pacioli studied the proportion and the relationship among numbers. Fibonacci is famous for discovering the Fibonacci sequence and related it to the Golden Ratio. Many artists during the Renaissance used the Divine Proportion to achieve balance and beauty in their paintings and sculptors. Many people believe that Leonardo da Vinci incorporated the Divine Proportion into *The Last Supper* and *Mona Lisa*. Even modern artists have used this ratio in their art. In the twentieth century, Mark Barr gave the ratio of the Divine Proportion the name Φ after the Greek mathematician and sculptor Phideas. The people listed are merely just a few among the many who contributed to discovering the Golden Ration and the many occurrences of it in nature (Hemenway, 2005, p.11-27).

There are multiple ways in which the Golden Ratio can be defined. Perhaps the simplest way or the most common definition is that of finding the golden section of a straight line. This problem was originally solved in Euclid's elements. Let a line AB of length l be divided into two segments by the point C. Let the lengths of AC and CB be a and b respectively. If C is a point such that $l:a$ as $a:b$, C is the "golden cut" or the golden section of AB (Huntly, 1970, p.25). To divide a straight line in the golden section, note the following:

"Let AB be the given straight line. Draw $BD=AB/2$ perpendicular to AB. Join AD. With center D, radius, DB, draw an arc cutting DA in E. With center A, radius AE, draw an arc cutting AB in C. Then C is the golden section of AB." (Huntly, 1970, p.27). Refer to the figure below:



Proof that *AC/CB* is the golden ratio is given below.

Proof: Let *AB=2x*. Then *DB=x*. By the Pythagorean theorem, *AD*=$\sqrt{4x^2 + x^2}$ = $x\sqrt{5}$. Thus,

AE = *AD* – *ED* = $x\sqrt{5}$ - x (since *ED* is the radius of the circle with center *D* radius *DB*)

AE = $x\sqrt{5} - x$

$AC = x\sqrt{5} - x$ (since AE and AC are both radii of the same circle).

So $CB = AB - AC = 2x - x\sqrt{5} + x = 3x - x\sqrt{5}$. Thus, $\frac{AB}{AC} = \frac{AC}{CB} \Rightarrow \frac{2x}{x\sqrt{5}-x} = \frac{x\sqrt{5}-x}{3x-x\sqrt{5}} \Rightarrow$

$6x^2 - 2x^2\sqrt{5} = 6x^2 - 2x^2\sqrt{5}$. Hence, *AC/CB* is the golden ratio.

Using the geometrical representation of the golden section, we can calculate the numerical the value of Phi. In the above figure, let *AC* = x, *CB* = 1, so that *AC/CB* = x = Φ. Note that

$$\frac{x+1}{x} = \frac{x}{1}, \text{ ie., } x^2 - x - 1 = 0.$$

The positive solution of this is $x = \frac{(1+\sqrt{5})}{2} = 1.61803$. Denote this as Φ (to 5 decimal places). Denote Φ′ to be the negative solution. If, instead of *CB* = 1, we take *AC* = 1 and *CB* = x′, then

$$\frac{x'+1}{1} = \frac{1}{x'}, \text{ i.e., } x'^2 + x' - 1 = 0.$$

The positive solution of this is $x' = \frac{(\sqrt{5}-1)}{2} = 0.61803$. This, prefixed by the negative sign, we call Φ′. So, Φ′ turns out to be the negative reciprocal of Φ; that is, Φ*Φ′ = - 1. For,

$$\frac{1}{\Phi} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2} = -\Phi'.$$

Phi is unique because it is the only number, which when diminished by unity, becomes its own reciprocal:

$$\Phi - 1 = \frac{1}{\Phi}, \text{ i.e., } \quad \Phi^2 - \Phi - 1 = 0.$$

Thus, Φ and Φ' are the solutions if $x^2 - 1 - 1 = 0$. Take Φ to be the positive solution $\frac{(1+\sqrt{5})}{2}$ and Φ' to be the negative solution $\frac{(1-\sqrt{5})}{2}$. Hence, $\Phi + \Phi' = 1$ and $\Phi * \Phi' = -1$ (Huntley, 1970, p. 26).

There are many properties of the golden section that make for an interesting discussion, and perhaps add to its appeal and fascination. The golden section as defined previously, is the proportion $\frac{(1+\sqrt{5})}{2} = 1.618\ldots$ In this case we are looking at the positive root of the equation $x^2 = x + 1$. Note the following tables:

Φ	1.618033989	Φ	1.618034
Φ^2	2.618033989	$\Phi+1$	2.618034
Φ^3	4.236067978	$\Phi+\Phi^2$	4.236068
Φ^4	6.854101966	$\Phi^2+\Phi^3$	6.854102
Φ^5	11.09016994	$\Phi^3+\Phi^4$	11.09017
Φ^6	17.94427191	$\Phi^4+\Phi^5$	17.94427
Φ^7	29.03444185	$\Phi^5+\Phi^6$	29.03444
Φ^8	46.97871376	$\Phi^6+\Phi^7$	46.97871
Φ^9	76.01315562	$\Phi^7+\Phi^8$	76.01316
Φ^{10}	122.9918694	$\Phi^8+\Phi^9$	122.9919
Φ^{11}	199.005025	$\Phi^9+\Phi^{10}$	199.005
Φ^{12}	321.9968944	$\Phi^{10}+\Phi^{11}$	321.9969

So by the table we can see that $\Phi^3 = \Phi^2 + \Phi^1$. In general, $\Phi^n = \Phi^{n-1} + \Phi^{n-2}$. In other words, each term of the sequence 1, $\Phi, \Phi^2, \Phi^3, \Phi^4, \ldots \Phi^n$ is the sum of the two previous terms. This sequence, often called the *Phi* series, is both additive and geometrical which is one of the reasons why it plays such a huge role in nature (Ghyka, 1977, p.8). The ratio of each term to the previous term is approximately equal to Φ. This property occurs in the Fibonocci sequence. Here is a list of the first few numbers in the Fibonocci sequence:

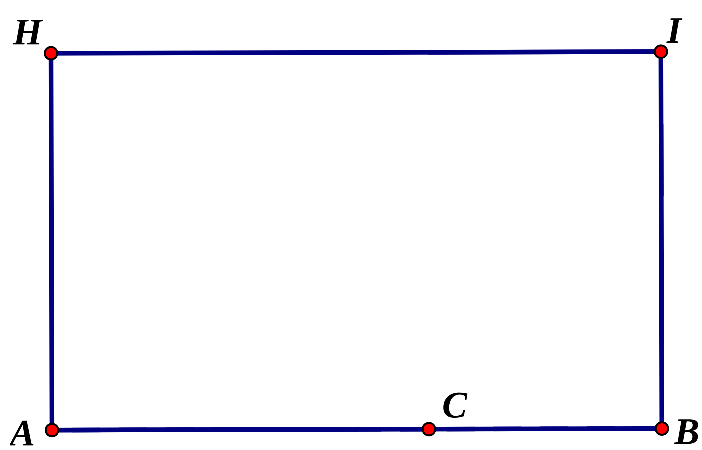
0,1,1,3,5,8,13,21,34,55,89,144,233,377…

Starting with the third term and looking at the ratios of each term to the previous term we have the following table:

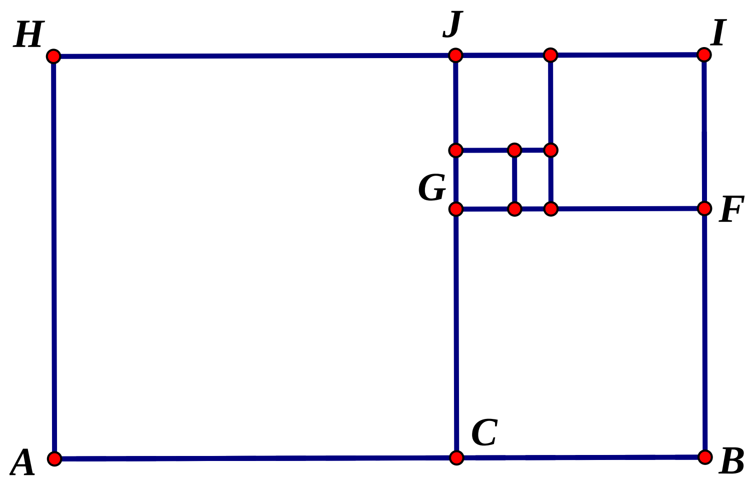
A	B	B/A
1	2	2
2	3	1.5
3	5	1.666667
5	8	1.6
8	13	1.625
13	21	1.615385
21	34	1.619048
34	55	1.617647
55	89	1.618182
89	144	1.617978
144	233	1.618056
233	377	1.618026
377	610	1.618037

As A and B get larger, B/A gets closer and closer to Phi. Hence, the Fibonocci sequence is closely related to the golden ratio.

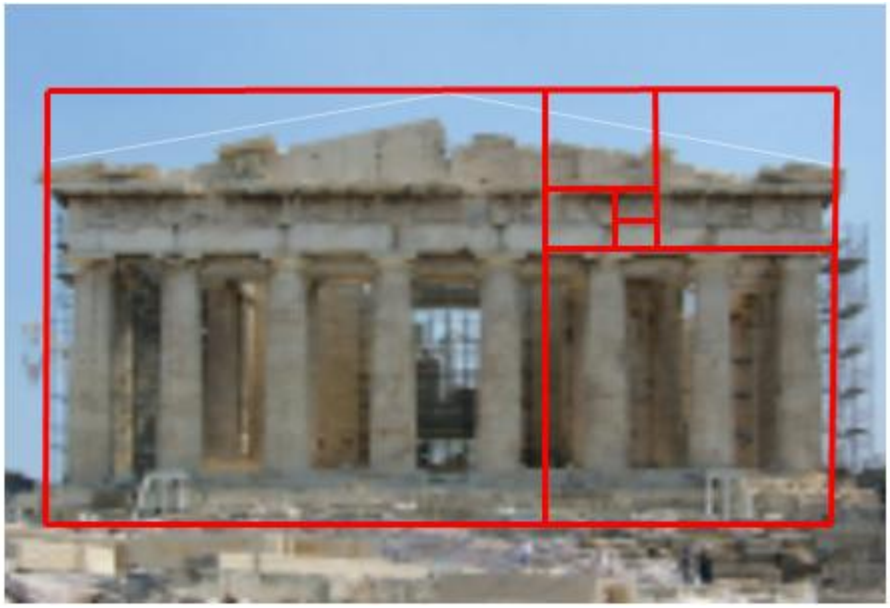
Referring to the figure of the construction of the golden cut from above, we can construct the golden rectangle. If we form a square with side *AC* on the line *AB* and complete the rectangle we obtain a golden rectangle *AHIB*.



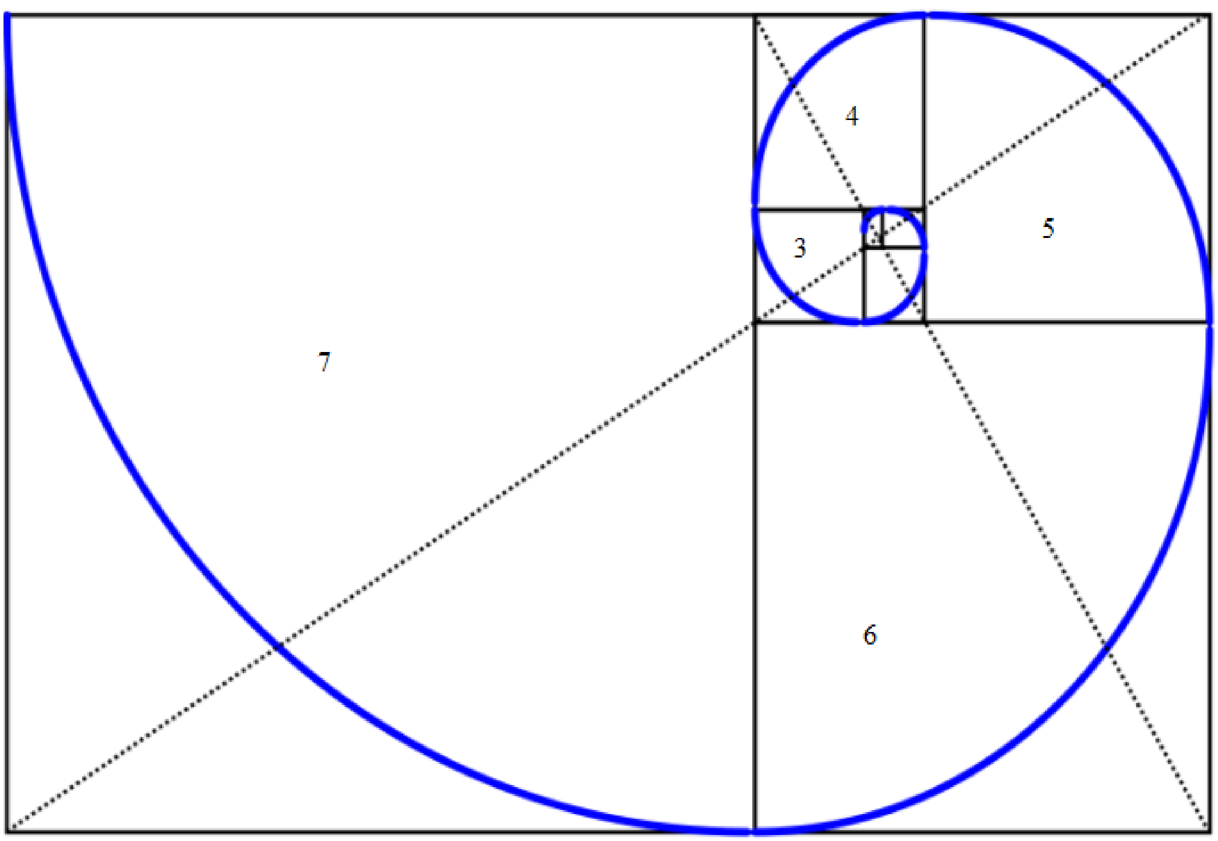
If we continue to construct squares on the smaller side of the golden cut, i.e. *CB*, we will always get another golden rectangle. This process can go on infinitely. For example, in the figure below, *GJIB* is a golden rectangle and *JGFI* is a golden rectangle, and so on.



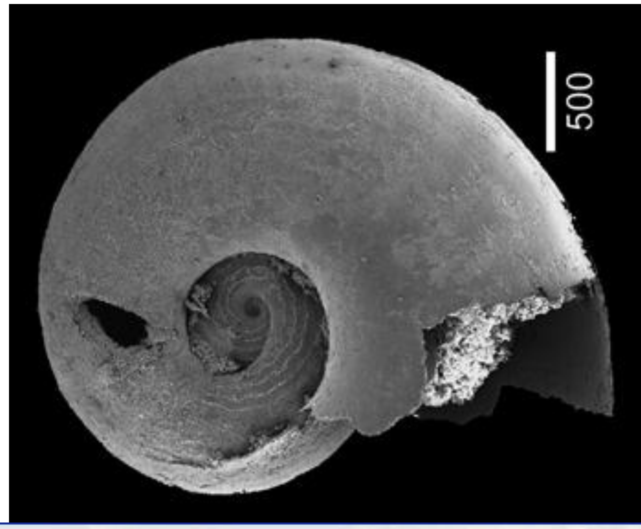
There are many examples of the golden rectangle in the real world. The Parthenon in Athens has dimensions that almost perfectly fit the golden rectangle’s dimensions. There is a lot of debate in which whether or not the dimensions of the golden rectangle is more aesthetically appealing to the eye. A German psychologist named Gustav Fechner did a lot of research on this matter. He made thousands of measurements of common rectangles that we see every day and discovered the average to be close to Phi. He measured books, playing cards, windows, etc. and came to the conclusion that most people prefer a certain rectangle design. These experiments were repeated by people who followed him and the results point to a popular preference for a rectangle with dimensions very close to those of the golden rectangle (Huntley, 1970, p. 63-65). As seen in the picture below, the dimensions of the Parthenon fit into a Golden Rectangle and its floor plan seems to be based on a square-root-of-5 rectangle (Hemenway, 2005, p.101).

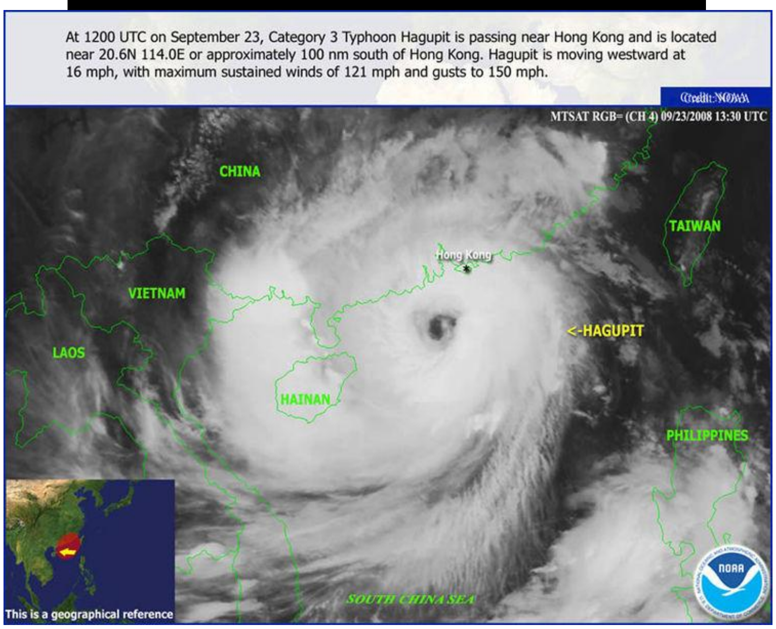


There is another way to construct the golden rectangle. As we saw previously, the golden ratio can be obtained from any addictive series. Similarly, the golden rectangle can be obtained from an additive series of squares.



Comparing the ratios of the sides we obtain: $\frac{(\sqrt{6}+\sqrt{7})^2}{(\sqrt{5}+\sqrt{4}+\sqrt{7})^2} \approx \frac{47}{21} \approx 1.620$. Continuing the process to square number 13, the approximation is $\frac{521}{322} = 1.6180$ ... (Huntley, 1970, p.66). Something that is also seen in the picture is that the centers of the squares lie on a spiral. This spiral, called the logarithmic or Golden Spiral, is connected with other constructions dealing with the golden ratio and is said to be appealing to the eye (Huntley, 1970, p.66). The logarithmic spiral is constructed by repeating the process of constructing golden rectangles within golden rectangles (as in the figure on page 6) until a limiting rectangle is reached. This limiting rectangle appears to be a single point and is called the pole of the logarithmic, or golden, spiral (Huntley, 1970, p. 101). The golden spiral passes through each golden cut formed in the construction which is one example of the connection between the golden spiral and the golden ratio. Another interesting property of the golden spiral is that no matter how different two segments of the spiral are in size, they are the same in shape. The lengths of the sides of the squares formed in the construction form the Fibonacci series. The relation between the Fibonacci series and the golden spiral is just another way to relate the Fibonacci series to nature. The spiral can be found in many things in nature (Huntley, 1970, p. 102). Some examples include the shell of a chambered nautilus, the oxygyrus (a free-floating oceanic snail), hurricanes, ocean patterns, and the cochlea of the human inner ear (Hemenway, 2005, p. 129-131).





The golden ratio is a very interesting topic of discussion in mathematics. There are many fascinating things in our world that contain the golden ratio. The golden ratio is a great way to show students that math relates to everyday life. Even our bodies contain the golden ratio! The best way to get students interested in what they are learning is to relate it to them somehow. A project involving exploration of the golden ratio and how it is represented in nature would be a great way for students to have fun and learn math at the same time.

References

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