Melissa Wilson EMAT 6450 - Mathematics in Context

**Course/Unit:** Accelerated Coordinate Algebra/Analytic Geometry A for Unit 9, Circles and Volume (This unit corresponds to Unit 3 in Analytic Geometry. The task can be modified with more scaffolding if needed.)

**Standards (GPS Common Core):**

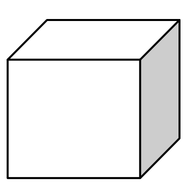
- MCC9-12.G.GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.
- MCC9-12.G.GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

**Description**: The task begins by facilitating a discussion on the difference between area and volume, and then moving into how the area volume formulas are derived. Students should have been exposed in previous courses to the area of rectangles/squares and triangles and to the volume of rectangular prisms. A review of these might be beneficial depending upon the particular group of students. See the overview of a discussion of the derivation of the formulas below.

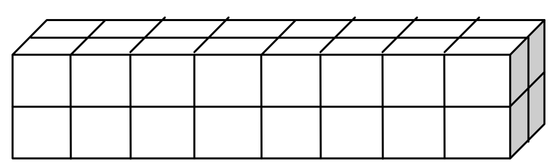
The "Wasted Space" task is intended to use the formulas in solving a real world problem. A good lead in to the task would be to present students with many different items in different packaging. The teacher should make sure to ask students why a company would choose certain packing for their items and what would be important to a company (aesthetics, price of packaging, transportation costs, etc.)

**Discussion for Derivation of Formulas:**

Begin by showing a cube that we will call the unit cube. This can be done by using a drawing or a model in class. Make sure that students understand that a unit cube has a side length of 1 unit and a volume of 1 unit$^3$ (same as a unit circle has a radius of 1). The construct a rectangular prism using unit cubes and ask students to find the volume. A diagram is shown below. Students might want to count the blocks, but be sure to guide students to finding a simpler method to finding the volume (V = base*height).

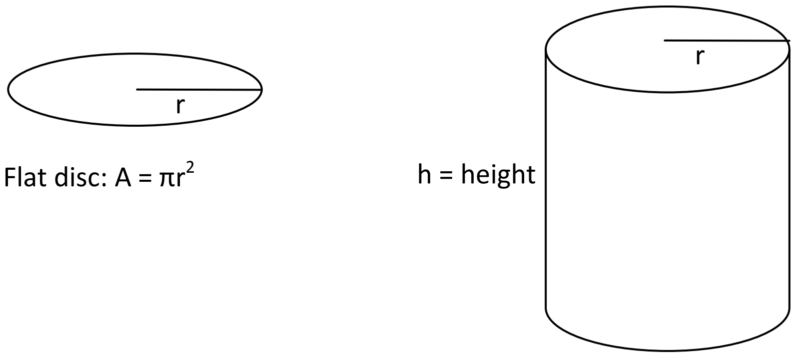


V = 1 u$^3$



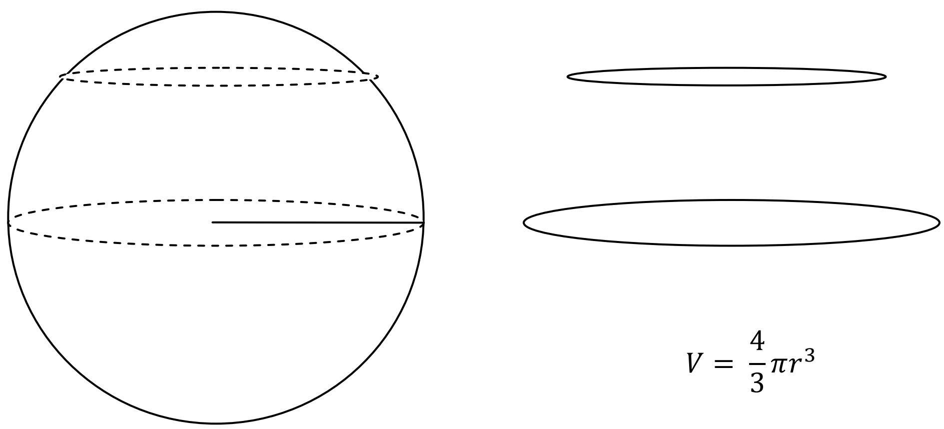
V = ?

This method can also be used to show the volume formula for a cylinder using an infinitely small disc:




$$V = \pi r^2 h$$

Lastly, the volume formula for a sphere can be shown as an analogy to the cylinder formula, only with the radius changing as well (see diagram below).  The formal limit argument is not the intention with this example since the discussion is intended for ninth graders.  However, it is important that students understand the infinitely thin discs have a varying radius:



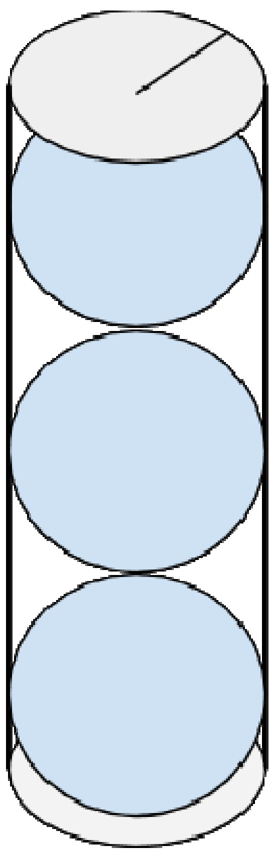
$$V = \frac{4}{3}\pi r^3$$

Cavalieri's principle is also mentioned as a method for deriving the volume formulas.  In essence, Cavalieri's principle states that if two objects have the same height and you cut both objects with a plane, and if the results cross-sections have the same area, then the volumes will be the same.  A great in class example is taking a stack of paper, books, or coins and finding the volume.  Then skew the stack and ask students about the volume.  Students should recognize that the volume has not changed, just shifted.  An excellencent resource on Cavalieri's principle is found on Wikipedia (http://en.wikipedia.org/wiki/Cavalieri's_principle).

***Wasted Space Task*** *(Answers/possible responses in **red** and extensions for the material in **purple**)*

Many factors affect the shape of packaging of items. What are some of the factors that you think affect the packaging a company may choose?
Students should arrive at several factors: size, number of objects included, price, shipping costs, aesthetics, etc.

For the following questions, consider three tennis balls packaged in a cylindrical canister. For the purpose of this problem, assume that the tennis balls touch the sides of the container and the top and bottom with no gaps.



1. How much wasted space (air) is there in this container? What would you need to know in order to answer this question? (For the purposes of this exercise, assume the bottom of the canister is flat instead of curved)
   Students should arrive at the conclusion: "wasted space" = volume of container - volume of objects inside.

2. What kinds of measurements would you need in order to answer this question?
   Teachers would know that the radius of each tennis ball (and the canister) is 1.4 inches, and the total height of the canister is 8.4 inches. Provide these measurements to students as they realize they are needed.

Extension: Let students solve the problem using variable of **r** for the radius and **6r** for the height.

3. Find the volume of the canister (round to the nearest hundredth):
   $V = \pi r^2 h$, where r = radius and h = height
   $V = \pi(1.4 \text{ in})^2(8.4 \text{ in})$
   $V = 51.72 \text{ in}^3$

4. Find the volume of an individual tennis ball (round to the nearest hundredth):

   $V = (4/3)\pi r^3$

   $V = (4/3)\pi(1.4 \text{ in})^3$

   $V = 11.49 \text{ in}^3$

5. Find the wasted space (round to nearest hundredth):

   $\text{Wasted space} = \text{Vol. of canister} - \text{Vol. of objects}$

   $\text{Wasted space} = (51.72 \text{ in}^3) - 3 * (11.49 \text{ in}^3)$

   $\text{Wasted space} = 17.25 \text{ in}^3$

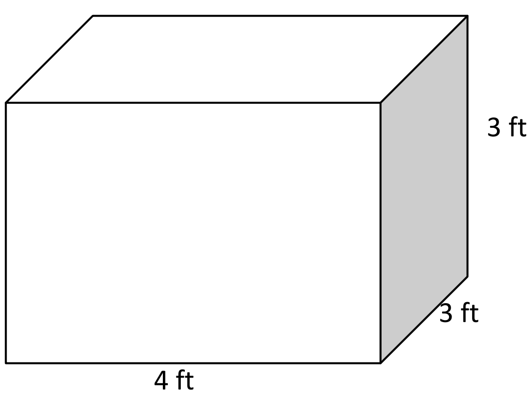
6. What is the percentage of the canister is wasted space?

   $(17.25 \text{ in}^3) / (51.72 \text{ in}^3) = 0.3335$

   $0.3335 * 100 = 33.35\%$

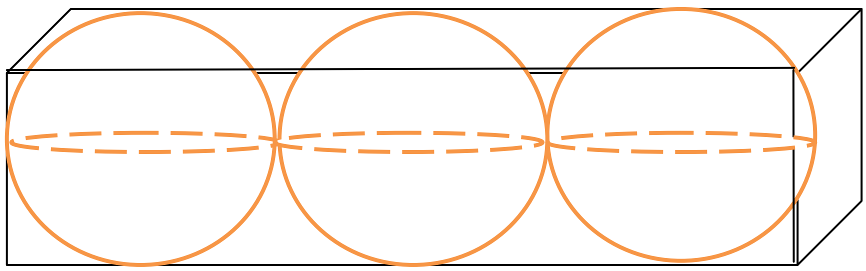
**Part 2**: This can be completed as an in-class activity or as a possible homework assignment. If done in class, you can provide students with several small objects.

1. Sometimes when packing a box to ship in the mail, the object does not always sit perfectly in a box touching all sides. You are to pick one object that can be modeled with several small shapes that can completely fit inside of the box shown (you must provide a picture or sketch of this object). In order to safely ship the object you must fill the box with packing peanuts (or some other similar product). Assume the packing peanuts are sold in bags of 15 $\text{feet}^3$ of material. How many bags would you need to ship the object below in the box shown? Show all work and formulas used.



2. (This can be used as an extension, extra assignment, or a warm-up activity the next lesson.) What if wanted to store three soccer balls (radius = 11 cm) in a box? What is the most efficient packing (i.e. least amount of wasted space) that you can find? Be sure to include a sketch of the packing, all work done, and the dimensions of the box you will use. (An extension would be to have the three soccer balls and also three basketballs (radius = 12cm) that you need to pack)

   One possible solution:



Volume of three soccer balls:

$$V = 3 * \left(\frac{4}{3}\pi(11cm)^3\right)$$
$$V = 16725.84\ cm^3$$

Volume of box:

$$V = (22\ cm) * (22\ cm) * (66\ cm)$$
$$V = 31944\ cm^3$$

Wasted space:

$$Wasted\ space = 31944\ cm^3 - 16725.84\ cm^3 = 15218.16\ cm^3$$

Percentage of empty space:

$$\frac{15218.16\ cm^3}{31944\ cm^3} = 0.4764$$
$$Pecentage = 47.64\%$$