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Situation: Solving Logarithmic Equations

Prompt:

Students in a high school classroom were asked to solve the following logarithmic equation: $\log_2(x-4) = 3-\log_2(x+3)$. The students solved the equation and got x = 5 and x = -4. The teacher told them that only the answer x = 5 was correct. A student asked, "I did all the math correctly! Why isn't x = -4 an answer as well?"

Commentary:

It is a common misconception in mathematics that if a correct algorithm is followed, only correct answers will result. This problem is an example of where that is not the case. These mathematical foci attempt to define logarithmic functions, properties of logarithms, and their uses. Graphs will be used to help explain why certain values of logarithms are undefined. Logarithms are often a difficult concept for students to grasp, and a teacher must have a basic understanding of this function in order to clear confusion.

Mathematical Foci:

Mathematical Focus 1

Logarithms are the inverses of exponents.

The logarithm of a number is the exponent to which another fixed value, called the base, must be raised to produce that number. For example, the logarithm of 1000 to base 10 is 3 because 10 to the power 3 is 1000: $10*10*10 = 10^3 = 1000$. In general, if $x = a^y$, then *y* is the logarithm of *x* to base *a* and is written $y = \log_a(x)$. The logarithm to base *a* = 10 is called the common logarithm. The natural logarithm is to base *e* and is denoted as *ln*. The basic idea to logarithms is to reverse exponentiation. For example, take the value 2^5. This is equivalent to 2*2*2*2*2 = 32. So in logarithmic notation we would write, $\log_2 32 = 5$. The base 2 raised to the 5^{th} power gives us 32. In general we do this for any positive real number base *a* not equal to 1 raised to any real number *y*. Note that when *a* is 1, any value *y* will still give us 1 because 1 raised to any power is still 1 so it does not make sense to use 1 as a base when dealing with logarithms. Secondly, we could use a negative base, but negative bases are only defined for certain values. For example, $\log_{-2}(8)$ is undefined because there is no real number y for which $(-2)^y = 8$. However, $\log_{-2}(-8) = 3$ since we know that $(-2)^3 = -8$. We can easily determine this for simple values such as these, but in general we do not use negative bases or take the logarithm of negative numbers because they are undefined more often than not. Also, when we have a positive base it does not make sense to take the logarithm of a negative number because no positive number raised to another positive number will give us a negative number. In order to solve for the logarithm of a negative number, we must use complex numbers. Thus for this context we define the logarithm of a negative number to be undefined and something that we cannot compute. When the base is negative and one is solving for the logarithm of a negative number, only odd powers will suffice. Thus, taking logarithms of negative numbers and using negative bases restricts our domain. This is also the same for 0 since no number (positive or negative) raised to any power will give you 0. Even though we don't use usually use negative bases or take the logarithm of a negative number, logarithms can be negative. For example, $\log_2(1/2) = -1$ because $2^{-1} = 1/2^1 = 1/2$. The following are some general conclusions that the above information gives rise to:

- The relationship $\log_a x = y$ is the same as $a^y = x$.
- Logarithms are really just exponents written in a different way.
- $\log_a a = 1$ for any base *a* because $a^1 = a$.
- $\log_a 1 = 0$ for any base *a* because $a^0 = 1$.
- $\log_a x$ is undefined if *x* is negative and *a* is positive. If *x* is negative and a is negative $\log_a x$ is defined for a strict domain of x.
- $\log_a 0$ is undefined for any base *a.*
- $\log_a a^n = n$ for any base *a.*

Mathematical Focus 2

Properties of logarithms

The following are some important properties of logarithms:

- The logarithm of a product is the sum of the logarithms of the numbers being multiplied: $\log_a(xy) = \log_a(x) + \log_a(y)$.
- The logarithm of a quotient is the difference of the logarithms of the numbers being divided: $\log_a(x/y) = \log_a(x)-\log_a(y)$.
- When a logarithm is being raised to a power *p*, the power *p* can be brought down in front of the logarithm: $\log_a(x^p) = p*\log_a(x)$.

As mentioned in the first focus, logarithms of base 10 are the common logarithm and logarithms of base e are natural logarithms. These are the only two logarithms that can be done in the calculator. As we have seen in the previous examples, there are some logarithms that can be easily calculated by hand. For others, we need to use the calculator. In order to use the calculator, all logarithms must be either base 10 or base e. This is not often what we are given to simplify. Thus we can use the change of base formula. The change of base formula converts a logarithm from a base that is not easily known to one that we can easily simplify or use the calculator for. The formula states the following:

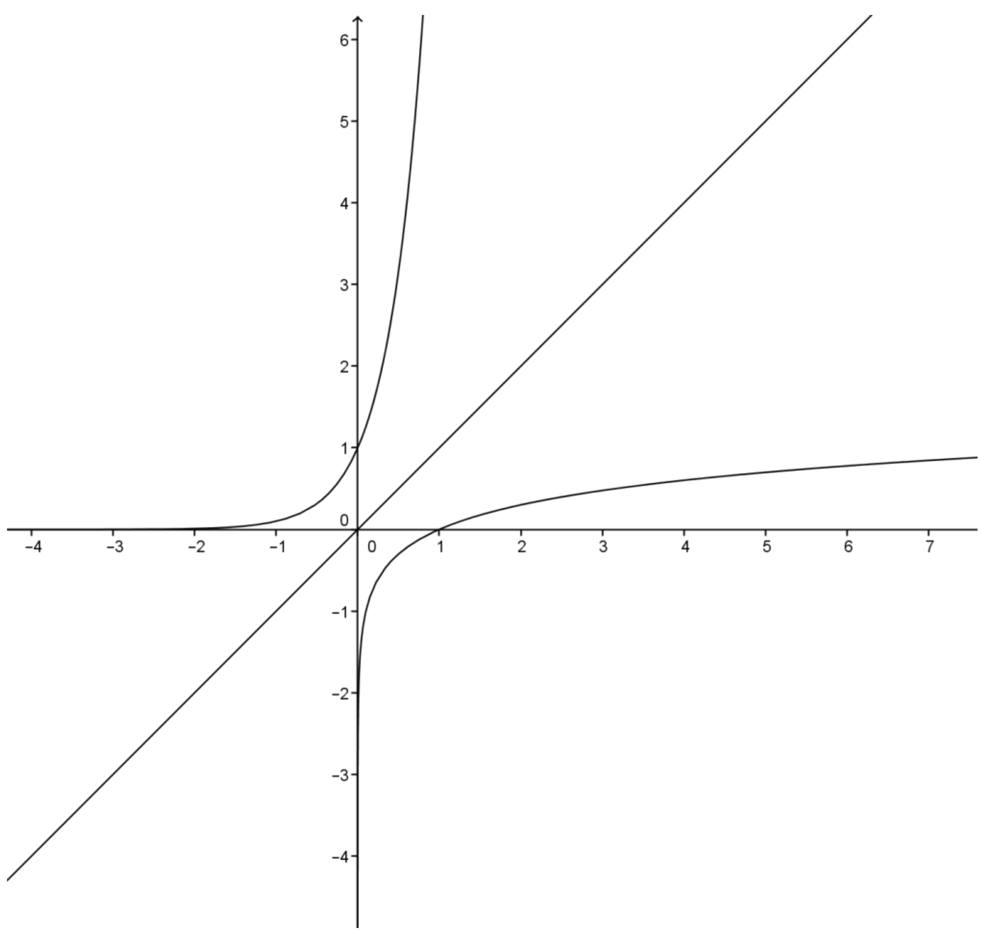
The logarithm $\log_a(x)$ can be computed from the logarithms of *x* and *a* with respect to an arbitrary base *k* using the following formula: $\log_a(x) = \log_k(x)/\log_k(a)$. If we pick *k* to be 10 or *e* then the logarithm can be solved using a calculator.

Note that log(x) represents the common logarithm with base 10 and ln(x) represents the natural logarithm with base *e*.

Mathematical Focus 3

Graphs of logarithmic and exponential functions.

A deep understanding of functions is good for acquiring a better understanding of logarithms. A function gives a rule for finding an output value given an input value. In this case we are looking at the function $y = a^x$. We want to show that this equation has a unique solution *x* for any positive *y* and positive base *a* not equal to 1. The unique solution *x* is the logarithm of *y* to base *a*, $\log_a y$. The function that assigns to y its logarithm is the logarithmic function. The graph of the logarithmic function y = log(x) is obtained by reflecting the function $y = 10^x$ across the line y=x.



As mentioned previously, the logarithmic function is the inverse of the exponential function and can be seen by the graph above. Note that the original function $y = 10^x$ is always positive for all values of x. The inverse function, $y = \log_{10}(x)$ is undefined for values of *x* that are less than or equal to 0. The domain for the original function is all real numbers, while the range is all real numbers greater than 0. The domain for the inverse function is all real numbers greater than zero while the range is all real numbers.

Mathematical Focus 4

Solving logarithmic equations.

Logarithmic equations involving two or more logarithms with the same base can be solved using the properties that were given above. If the logarithms do not all have the same base, the change of base formula must be used. When the logarithms have the same base then the equation can be solved using a simple algorithm. As an example, we will work with the equation that was brought up in the prompt:

$$\log_2 (x-4) = 3-\log_2 (x+3)$$

$$\log_2(x-4) + \log_2 (x+3) = 3$$

$$\log_2[(x-4)(x+3)] = 3$$

$$2^{\log_2 (x-4)(x+3)} = 2^3$$

$$(x-4)(x+3) = 8$$

$$x^2-x-12 = 8$$

$$x^2-x-20 = 0$$

$$(x-5)(x+4) = 0$$

$$x = 5, x = -4$$

Now we have to plug both of the values of x back into our original equation to determine the solution. Note that when 5 is substituted back in we obtain $\log_2(1) = 3 - \log_2(8)$ which is valid. However, when we plug -4 in, we do not get a valid solution because we cannot take the logarithm of a negative number. Thus, the solution to the original equation is x = 5.

<u>Post commentary:</u>

Logarithms are often a struggle for students. If we can simply explain logarithms in terms of exponents then I think that students would have a much easier time with the concept. Looking at the graphs of exponential functions and its logarithmic inverse function can really help students visualize why we cannot take the logarithm of a negative number, and hence why we must check our solutions in the original equation when solving logarithmic equations.

The invention of logarithms is credited to John Napier. He defined logarithms as a ratio between two distances as opposed to the current definition of exponents. His original idea was to find a way to simplify calculations with really large numbers. Napier constructed an entire table of logarithm estimations by finding a number whose logarithm possesses upper and lower bounds that differ by an insignificant amount. He then reasoned the average of these bounds was a good estimate of the actual logarithm. This is related to the area under a curve. Thus, another definition of the natural logarithm is as follows: $\ln(x) = \int_1^x \frac{1}{t} dt , x > 0$.