Situation 1
July 8, 2013

1 Prompt:

A calculus class was well into a unit on sequences and series. The teacher just introduced the idea of taking the limit of a series to determine convergence when a student asked "What is the difference between the limit and the sum of a series?"

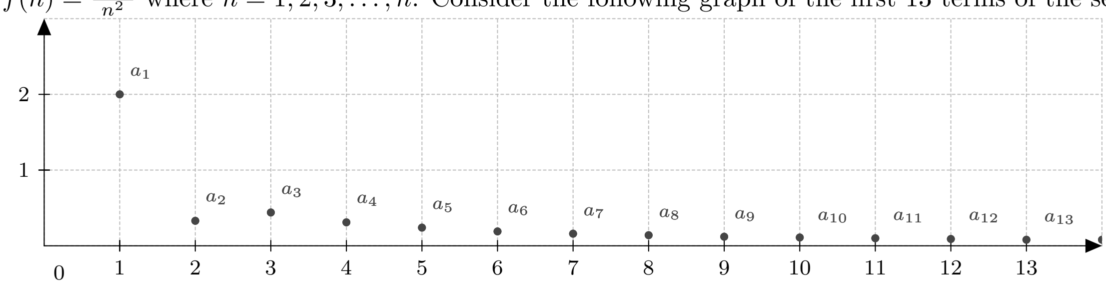
2 Commentary

Sequences and series is often a difficult topic for students. It is easy to get sequences and series confused since a lot of the same terminology is used for both. A sequence is a list of numbers while an infinite series is a sum. Therefore, if the sum of a series exists, it will be a single value. It is important that teachers understand the concepts behind sequences and series, and the difference between taking the limit of a series versus evaluating the sum of a series. This set of mathematical foci will address definitions, limits, convergence, divergence, and will provide examples to demonstrate the diffeence between the limit and the sum of a series.

3 Mathematical Foci

Focus 1: Sequences

A sequence is a set of numbers or terms in a specific order. The length of a sequence can be finite or infinite. An example of a finite sequence is $\{1, 2, 3, 4, 5\}$. The set of natural numbers is an example of an infinite sequence. General sequence terms are denoted as follows: a_1, a_2 ,... a_n, a_{n+1}, ... where $a_1 =$ the first term, $a_2 =$ the second term, $a_n =$ the n^{th} term, and $a_{n-1} =$ the $(n+1)$ term. Each of the following are equivalent ways of denoting a sequence: a_n, $\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\}$, or $\{a_n\}_{n=1}^{\infty}$. It is important to note that the first term of a sequence does not have to start at 1. The third example implies that the sequence starts at 1, but it can start anywhere. The formula for a sequence can be defined by a recursive formula in which each term is used to produce the next term, or it can be defined using an explicit formula which is a rule that produces the n^{th} term when plugging in that specific value for n.
Consider a sequence in which there is a defined recusrsive formula for. We can essentially treat this formula as a function that can only have integers plugged into it. For example, the sequence $a_n = \frac{n+1}{n^2}$ can be treated as $f(n) = \frac{n+1}{n^2}$ where $n = 1, 2, 3, \ldots, n$. Consider the following graph of the first 13 terms of the sequence:



This leads to an important property of sequenes. Note that as n gets larger and larger, the terms of the sequence seem to approach 0. Thus, we say that the **limit** of the terms of the sequence is 0. We denote this as $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$.

Definition. *Limit of a Sequence We say that* $\lim_{n\to\infty} a_n = L$ *if for every number* $\epsilon > 0$ *there is an integer* N *such that* $|a_n - L| < \epsilon$ *whenever* $n > N$*. We say that* $\lim_{n\to\infty} a_n = \infty$ *if for every number* $M > 0$ *there is an integer* N *sucht that* $a_n > M$ *wherever* $n > N$*. We say that* $\lim_{n\to\infty} a_n = -\infty$ *if for every number* $M < 0$ *there is an integer* N *such that* $a_n < M$ *wherever* $n > N$*.*

Now, if the $\lim_{x\to\infty} a_n$ exists and it is finite, we say that the sequence **converges**.If $\lim_{x\to\infty} a_n$ doesn't exist or is infinite we say that the sequence **diverges**. To find the limit of a sequence in most cases, we just treat the limit as if it were a limit of a function. For example, consider the sequence $\{a_n\}_{n=1}^{\infty} = \frac{3n^2-1}{10n+5n^2}$.

To evaluate the limit note the following:

$$\lim_{n\to\infty} \frac{3n^2-1}{10n+5n^2} = \lim_{n\to\infty} \frac{n^2(3-\frac{1}{n^2})}{n^2(\frac{10}{n}+5)} = \lim_{n\to\infty} \frac{3-\frac{1}{n^2}}{\frac{10}{n}+5} = \frac{3}{5}.$$

So the sequence converges and its limit is $\frac{3}{5}$.

Focus 2: Series

A series is the sum of a finite or infinite sequence. Given an infinite sequence $\{a_n\}$, a **series** is informally the result of adding all those terms together: $\{a_1+a_2+a_3+\cdots\}$. Series are usually expressed using summation notation. An example of an infinite series is the following: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. The terms of the series are often produced according to a certain rule, such as by a formula or an algorithm. Unlike finite summations, infinite series need tools from mathematical analysis, and specifically the notion of limits, in order to be fully understood and evaluated. When discussing an infinite series, we often refer to the sequence of partial sums defined as the following:

Definition. *The sequence of partial sums $\{S_k\}$ of an infinite series $\sum_{n=1}^{\infty} a_n$ is defined for each k as the sum of the sequence $\{a_n\}$ from a_1 to a_k*

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + \cdots + a_k.$$

By definition, the series $\sum_{n=1}^{k} a_n$ **converges** to a limit L if and only if the associated sequence of partial sums $\{S_k\}$ converges to L. If the limit of S_n is infinite or does not exist, the series is said to diverge. When the limit of the partial sums exists, it is called the sum of the series. In general finding a formula for the general term in the sequence of partial sums is a very difficult process. Thus, finding the value or sum of a series is equally difficult since the general formula for partial sums is required. For example, consider the following sequence: $\sum_{n=1}^{\infty} n$. Intuitively, we note that the series looks like this: $1+2+3+4+5+\cdots+n$. Since n is going to infinity, we can imagine that the sum is also going to go to infinity since we are adding up an infinite amount of numbers that are getting infinitely larger. To dertermine convergence technically, we first need a formula for the general term in the sequence of partial sums. This is a known series and its value can be shown to be: $s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. To determine if the series is convergent we will need to see if the sequence of partial sums $\{\frac{n(n+1)}{2}\}_{n=1}^{\infty}$ is convergent of divergent. Taking the limit we deduce, $\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$. Therefore the sequence of partial sums diverges to ∞ and so the series also diverges. It is important to note the difference between the series, and the formula for the general term in the sequence of partial sums. Take this series, $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. If we just take the limit of $\frac{1}{n^2-1}$, we deduce that the terms of the sequence that this series represents converges to 0. However, if we find a formula for the partial sums, we can compute the sum and it is not equal to 0. Note that

$$s_n = \sum_{i=2}^{n} \frac{1}{i^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

and in this case we have,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (\frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}) = \frac{3}{4}.$$

The sequence of partial sums converges and so the series converges and its value is, $\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$.

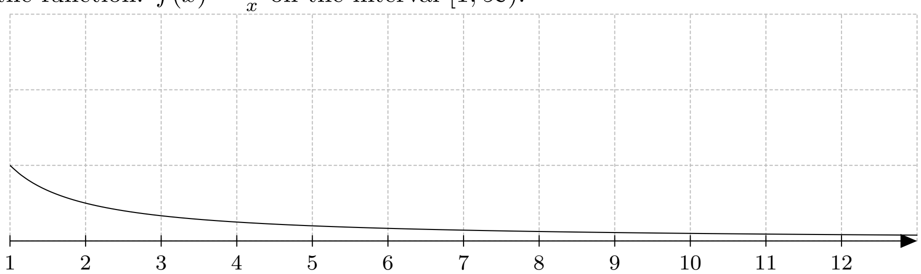
Focus 3: The Geometric and Harmonic Series

There are a few special types of series that we study that are important to understanding this prompt. A **geometric series** is a series in where each successive term is generated by multiplying the previous term by a constant factor. This constant factor is called the common ratio r. The general form of a geometric series is $\sum_{n=1}^{\infty} ar^{n-1}$ where $a \neq 0$. A geometric series is convergent if $|r| < 1$, and divergent otherwise. The sum of the first n terms of a geometric series is $s_n = \frac{a(1-r^n}{1-r}$. The sum of the series $\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$. An example of a geometric series is: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$. Here, the common ratio r is $1+\frac{1}{2}$. Since $\frac{1}{2} < 1$ we deduce that the series is convergent and the sum is $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$ Note that the formula for the sum of an infinite geometric series is the limit of the partial sums. The limit of the partial sums goes to 2 so we say that the sum of the series is 2. This does not mean that the terms of the sequence converge to 2. Clearly the terms of the sequence are converging to 0 as n approaches infinity. We are only able to determine the sum of a geometric series because there is a formula for the partial sum.

Let's take a look at another series, the **harmonic series**. The harmonic series is a famous series that is divergent. The harmonic series is defined as: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. This is counterintuitive since the limit of the nth term as n goes to infinity is zero. One would think that the limit of partial sums would converge to a number, however it can be proven that the sum of the harmonic series diverges to infinity. One was to prove this is to compare the harmonic series to another divergent series. Note the following: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots$. Each term of the harmonic series is greater than or equal to the corresponding term of the other series. We can show that the sum of the second series is infinite: $1 + (\frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + (\frac{1}{16} + \cdots + \frac{1}{16}) + \cdots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$. By the comparison test, the harmonic series is divergent must also diverge as well.

Focus 4: Tests for Convergence

In the previous foci we used the comparison test to show that the harmonic series is divergent. The comparison test states: Suppose there are two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then, if $\sum b_n$ is convergent then so is $\sum a_n$ and if $\sum a_n$ is divergent then so is $\sum b_n$. The integral test is also useful to prove that the harmonic series is divergent. The integral test states: If $f(x)$ is a continuous, positive and decreasing function on the interval $[k, \infty)$ and $f(n) = a_n$ then, if the integral from k to infinity of $f(x)$ is convergent then so is $\sum_{n=k}^{\infty} a_n$ and if the integral from k to infinity of $f(x)$ is divergent then so is $\sum_{n=k}^{\infty} a_n$. To show that the harmonic series is divergent using the integral test, let's look at the graph of the function: $f(x) = \frac{1}{x}$ on the interval $[1, \infty)$.



Evaluating this improper integral from one to infinity we deduce that the integral diverges. Thus the harmonic series diverges by the integral test. There are other tests we use to determine whether or not a series converges or diverges. Two tests that require taking a limit are the ratio test and the root test. The **ratio test** states that for a series $\sum a_n$, define $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. If $L < 1$ the the series is convergent, if $l > 1$ the series is divergent, and if $L = 1$ then the ratio test is inconclusive. The **root test** states, that for a series $\sum a_n$, Define $L = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$. If $L < 1$ the series is convergent, if $L > 1$ the series is divergent, and if $L = 1$ the root test is inconclusive. These limits only allow us to determine convergence. This limit tells us nothing about the value of the sum.

4 Post-Commentary

These mathematical foci attempt to shed light on the difference between the value of the sum of a infinte series, and the limit of the terms of the sequence that the series describes. As shown with the harmonic series, just because a the terms of a series converge to a certain number, does not necessarily mean that the sum of the terms will also converge. Calculating the sum of a series is often times a very difficult task that is beyond the scope of the classroom in which this prompt was brought up. Certain series such as geometric series and teloscopic series provide relatively simple ways to calculate the sum. For other series, a formula for the partial sum must be derived. It is important to understand and recognize the difference between the limits and sums. There are tests for convergence that require taking a limit, but do not tell us what the value of the sum is.