## **Situation: Adding Fractions**

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# **Prompt**

A teacher is working through a lesson on the rules of exponents. She begins with the addition property of exponents with the same base, i.e.  $(x^m)(x^n) = x^{m+n}$ , where  $m, n \in \mathbb{R}$ . The students appear to understand the concept of adding exponents, until the teacher worked through an example where m and n are fractions:  $(x^{\frac{1}{2}})(x^{\frac{4}{5}}) = x^{\frac{1}{2}+\frac{4}{5}} = x^{\frac{13}{10}}$ . Many of the students piped up and asked, "After adding  $\frac{1}{2} + \frac{4}{5}$  isn't the exponent  $\frac{5}{7}$ ? How did you get  $\frac{13}{10}$ ?"

### **Commentary**

Many students have forgotten basic algebraic skills due to lack of practice, insufficient teaching, calculators, etc. Adding fractions is a great example of how students jump to a solution or process rather than think of the mathematics they are attempting to perform. This situation is designed to refresh teachers' memories of the process and mathematics of adding fractions. Through the following explanations and visuals teachers may better explain and solidify their students' understanding of adding fractions. The following foci include discussions on rational numbers, divisors, dividends, and visual representations.

# **Mathematical Foci**

### Focus 1: Rational Numbers and Irrational Numbers

A rational number is a number that can be written as a simple fraction  $\frac{m}{n}$  where *m* and *n* are integers and *n* does not equal zero. Recall that all integers can be written as a rational number because *n* can be one. Therefore, rational numbers are a subset of the real number system, i.e.  $\mathbb{Q} \subset \mathbb{R}$ . The set of rational numbers is closed under all four operations. This means that the sum, difference, product, and quotient of any two rational numbers will produce a rational number. This property holds as long as *n* does not equal zero (to see more information on dividing by zero refer to the following webpage:

http://jwilson.coe.uga.edu/EMAT6500/SIT46/Sitn%2046%20DivbyZero%20071120.pdf)

The properties of rational numbers are as follows:

- 1.  $\mathbb{Q}$  is an ordered field.
- 2.  $\mathbb{Q}$  is Archimedean (given a number you can always find one larger than the first)
- 3.  $\mathbb{Q}$  is countable to infinity

4. Q is not complete (there are bounded increasing sequences of rationals, which have no limit in Q).

\* $\sqrt{3}$  ∉  $\mathbb{Q}$  ⇒ $\sqrt{3}$  = 1.732.. is the limit of the sequence 1, 1.7, 1.73, 1.732...

Proof of Part 4: Suppose to the contrary that  $\sqrt{3} \in \mathbb{Q}$ . Then we could write  $\sqrt{3} = \frac{m}{n}$  where m,  $n \in \mathbb{N}$  with  $n \ge 1$  and gcd(m, n) = 1. Hence  $n\sqrt{3} = m$ . Thus given  $n^2 \times 3 = m^2$ . But 3 is a prime and  $3|3n^2$  so  $3|m^2$  so  $3|m \times m$ . Since 3 is prime we conclude that 3|m. Hence m = 3M for some integer M. This implies  $3n^2 = m^2 = (3m)^2 = 9M^2$  so  $n^2 = 3M^2$ . This implies that  $3|n^2$  hence  $3|m^2$  which is a contradiction with the assumption that the gcd(m, n) = 1. Therefore,  $\sqrt{3} \notin \mathbb{Q}$ .  $\Box$  (Rumely, 2013, n.p).

Additionally, rational numbers will either have terminating decimals or recurring decimals. For example,  $\frac{435}{100} = 4.35$  the number terminates and for  $\frac{3}{7} = .428571$  the numbers are periodic just after the decimal, i.e. the numbers after the decimal infinitely repeat. Converting a rational number written as a fraction into a decimal can be observed through long division. Let's look at the example  $\frac{3}{7}$ .

By dividing the integer 7 into 3 we are able to obtain the recurring decimal .428571. This process of long division could continue infinitely many times and the numbers will continue to repeat. This repetition continues because the remainder will never equal zero. If the rational number has terminating decimals then the following would occur. Follow the example  $\frac{7}{8}$  below.



Notice, when finding the decimal representation of the rational number  $\frac{7}{8}$  through long division the process ends when zero occurs as the remainder. Thus, the numbers behind the decimal terminate.

An irrational number is a number that cannot be written as a simple fraction  $\frac{m}{n}$  where *m* and *n* are integers and *n* does not equal zero. Irrational numbers are also a subset of the real number system, i.e.  $\mathbb{M} \subset \mathbb{R}$ . Let  $\mathbb{M}$  represent the set of irrational numbers. The key to this definition is that *m* and *n* are integers. This is important to realize for of course the following can occur:  $\frac{\sqrt{2}}{1} \approx 1.414213562$  ...The previous fraction may look rational, but in fact it is not for one  $\sqrt{2}$  is not an integer and the decimals do not terminate or repeat. Thus it is an irrational number.

Discussing both the rational and irrational numbers is necessary for as stated in the prompt  $m, n \in \mathbb{R}$  in the problem  $(x^m)(x^n) = x^{m+n}$ . The domain of *m* and *n* is limited to the real number system.

#### **Additional Side Proof:**

Let  $0.\overline{89}$  be the repetend you can show this is the repetend of a rational number through the following strategy:

Let  $100s = 89.\overline{89}$  and  $s = 0.\overline{89}$  then  $100s - s = 99s = 89.\overline{89} - 0.\overline{89} = 89$  concluding that  $s = \frac{89}{99}$ , which is a rational number.

Thus, a rational number exists if and only if there is a repeating decimal.

#### Focus 2: Denominators and Numerators

Finding a common denominator is a key concept to understanding the addition of fractions. This process becomes routine and robotic for teachers. The true understanding of finding a common denominator is lost with the drill. Though adding fractions is a fairly simple concept we as mathematicians need to also be thinking about the mathematics behind the process while making calculations. Common denominators are necessary to find for each fraction must be written as a multiple of a common fraction.

Looking back at the prompt and applying this idea we have the following:

$$\frac{1}{2} + \frac{4}{5} = \frac{1(5)}{2(5)} + \frac{4(2)}{5(2)} = \frac{5}{10} + \frac{8}{10} = 5 \times \frac{1}{10} + 8 \times \frac{1}{10}$$

Thus,  $\frac{1}{10}$  is our common multiplier. By finding the common multiplier our  $\frac{5}{10} + \frac{8}{10}$  fractions are now in the same terms. It is not necessary to find the least common multiple of the denominators, but multiplying by the least common multiple will keep the fractions smaller.

A general form of the concept above is as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} = ad \times \frac{1}{bd} + cb \times \frac{1}{bd} = \frac{ad + cb}{bd}$$

The numerator of a fraction goes along with the ride. When the denominator is multiplied by a number the numerator is multiplied by that same number in order to maintain the same proportion. The fraction is not changed because anything (excluding zero) multiplied by one is itself. Remember the fractions are numerical quantities; they are not just one integer stacked upon another integer.

#### Focus 3: Visual for Adding Fractions

Many teachers are visual learners, but even those who do not classify themselves as visual learners may find the following focus beneficial for understanding the breakdown of adding fractions. Looking at the two rectangles below, notice both are of equal area. This equal area will be our unit of measurement.



The rectangles above have been created to represent  $\frac{1}{2} + \frac{4}{5}$ . From here on rectangle A will represent the rectangle divided into halves and rectangle B will represent the rectangle divided into five equal pieces. To clarify notice rectangle A has been divided into two equal halves shaded with two different colors. Notice rectangle B has been divided into five equal pieces where four pieces are shaded dark pink. The visual is saying in order to add  $\frac{1}{2} + \frac{4}{5}$  we must add the blue half with the four dark pink pieces. But how can this happen when the pieces are of different sizes? We must find a common denominator. See Focus 2 for why finding a common denominator is necessary. Observe the new visual below.



Each whole rectangle has been divided into ten equal pieces. Ten is the least common multiple of 2 and 5. After dividing the rectangles into ten equal pieces rectangles A and B are now in the same playing field. The two visuals above show that finding a common denominator for each fraction does not affect proportion. The same area is shaded for each triangle despite the number of divisions. Now looking at the visual above we have the following:  $\frac{5}{10} + \frac{8}{10} = \frac{13}{10}$ . Notice there are five blue pieces shades and eight dark pink pieces shaded. Adding the blue and pink pieces we have a total of thirteen pieces, which gives us the numerator of our final fraction. Therefore, our final solution leaves thirteen pieces each  $\frac{1}{10}$  of a rectangle.



#### Focus 4: Extension- A Look into Multiplying Fractions

How can we find  $\frac{2}{3}$  of  $\frac{4}{5}$  of a candy bar? Let's first write out our equation by breaking up the fractions to see if this helps us visualize how we will divide this Snickers Bar seen below.

$$\frac{2}{3} \times \frac{4}{5} (\text{Snickers Bar})$$
$$= 2(\frac{1}{3}) \times \frac{4}{5} (\text{Snickers Bar})$$
$$= 2(\frac{1}{3}) \times 4(\frac{1}{5}) (\text{Snickers Bar})$$
$$= 2 \times 4 \times (\frac{1}{3}) \times (\frac{1}{5}) (\text{Snickers Bar})$$
$$= 8(\frac{1}{15}) (\text{Snickers Bar})$$

Our unit of measurement is the one Snickers Bar. So to find out how many total slices we have we will look at the Snickers Bar as a whole. As a whole the candy bar has been sliced into 15 even pieces. This number was found by first dividing our candy bar into 5 even pieces and then dividing those 5 pieces individually 3 more times. So we are left with 15 smaller pieces. We know we have divided correctly, because our denominator is equal to 15. Next we will look at

how many  $\frac{1}{15}$  pieces we need. Our equation asks for 4. So in the model we have highlighted 4 purple pieces each equaling  $\frac{1}{15}$  of the entire candy bar. Now how many  $4(\frac{1}{15})$  do we need? Our equation asks for 2 so we have doubled our portion and are left with 8 slices each equal to  $\frac{1}{15}$  (Snickers Bar). It is very important for students to understand multiplication of fractions. Dividing a candy bar according to the proportions given shows students that dividing this Snickers Bar is far more in depth than taking  $\frac{8}{15}$  of a candy bar. By breaking down the fractions, students see each individual piece of the equation.



#### **Legend of Pythagoras:**

Legend has it that more than two thousand years ago the Greek mathematician Pythagoras of Samos believed that all numbers were rational. Pythagoras was the leader of the prestigious mathematics cult in Greece. One day one of his students contradicted Pythagoras theory of all numbers being rational. The student supposedly used geometry to show Pythagoras that the  $\frac{\sqrt{2}}{1}$  could not be written as a rational number. Unfortunately this moment of audacity cost the poor student his life for he was tossed into the sea and drowned (Rational Numbers, n.d., n.p.).

### **Post Commentary**

Adding fractions is an elementary concept, but unfortunately many students do not understand this concept and teachers are forced to reteach old material. Whatever the reasons for this lack of comprehension among students, teachers need to be prepared to reteach the concept with the goal of having the concept stick. Mathematician, Richard Askey, observed the following interesting comparison,

The concept of fractions as well as the operations with fractions taught in China and the U.S. seem different. U.S. teachers tend to deal with "real" and "concrete" wholes (usually circular or rectangular shapes) and their fractions. Although Chinese teachers also use these shapes when they introduce the concept of a fraction, when they teach operations with fractions they tend to use "abstract" and "invisible" wholes (e.g., the length of a particular stretch of road, the length of time it takes to complete a task...) (Askey, 1999, p. 7).

Askey observed that teachers in the U.S. tend to shy away from "abstract" wholes and instead teach students the concept of fractions using standard shapes. Unfortunately, this method may be working against the teachers for presenting material such as adding fractions abstractly may help further solidify the content within the students. This would require additional preparation, understanding, and determination, but it may be the key to keeping students from falling into the hole of misconceptions with fractions. In conclusion, the foci discussed above were designed to refresh teachers on the concepts of adding fractions, and the comments discussed in the post commentary were added to encourage teachers to think and teach outside of circles and rectangular shapes.

#### References

Askey, R. (1999). Knowing and Teaching Elementary Mathematics. *American Educator*, American Federation of Teachers.

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Rational Numbers. (n.d.). *Math is Fun - Maths Resources*. Retrieved July 11, 2013, from <u>http://www.mathsisfun.com/rational-numbers.html</u>