# Situation: Infinity + Infinity

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**Prompt** : **When discussing large numbers and properties of the addition, a student asks, "What is infinity plus infinity?"**

**Commentary**

Firstly, addition is an operation only for numbers, which $\infty$, in the traditional sense is *not*. We must look to the arithmetic of cardinal numbers to make sense of $\infty + \infty$. Further, when discussing infinity, it is imperative to know *what* "infinity" we are talking about. For any grade school student, the infinity at hand is $\infty$, or $\aleph_0$ - the countable infinity that comes as a result of considering which "number" comes after all the natural numbers, or equivalently how many natural numbers there are. For this $\infty$, it is intuitive that $\infty + \infty = \infty$. For both the case of infinity as $\infty = \aleph_0$ and other cases, we will use cardinal arithmetic to sum infinity with itself.

**Mathematical Foci**

**Focus 1** : *Useful definitions & concepts for a set theory approach to infinity*

Symbols & Notation

$\mathbb{N}$ - the set of all natural numbers $\{1, 2, 3, ...\}$.

$\mathbb{R}$ - the set of all real numbers

$\in$ - is an element of, i.e. $3 \in \mathbb{N}$ is synonymous with "3 is an element of set of the natural numbers" and "3 is a natural number."

$\notin$ - is not an element of, i.e. $\sqrt{-1} = i \notin \mathbb{R}$.

$\subset$ - is a subset of, i.e. $\mathbb{N} \subset \mathbb{R}$.

Set notation & definition - A set is here defined as a group of objects that fit a certain definition or description and is notated by $A = \{x \,|\, P(x)\}$. This reads as "the set $A$ is defined as all elements $x$ such that $x$ satisfy property/definition $P$.

- Ex. $\{x \,|\, x = 2n, n \in \mathbb{N}\}$ is the set of all $x$ such that $x$ equals a natural number multiplied by two, i.e. the set of all positive even numbers.

$\Rightarrow$ - implies, i.e. $p \Rightarrow q$ is synonymous with "Statement $p$ implies statement $q$," and "$2x = 6 \Rightarrow x = 3$ is synonymous with "$2x = 6$ implies $x = 3$."

$(A \to B)$ - $A$ maps to $B$. The arrow designates a function with domain of $A$ and co-domain

of $B$. The co-domain is only equal to the range if the function is surjective (see below).
$\emptyset$ - the empty set, i.e. a set with no elements. (Note: this is still a set; it is not "nothing".)

Definitions

Injective - A function $f(x)$ is *injective* if it is one-to-one, i.e. if every $x$ in the domain maps to a distinct $y$ in the co-domain (i.e. no two $x$ values can map to the same $y$ value). Formally, $f$ is injective if $f(x) = f(y) \Rightarrow x = y$. An injective function $f$ is called an *injection*.

- Ex. f(x) = 2x is injective, because all even numbers are distinct.

Surjective - A function $f(x) : A \to B$ is *surjective* if every element $y$ in $B$ is mapped to by some element $x$ in $A$. Another word for surjective is *onto*. A surjective function $f$ is called a *surjection*.

- Ex. f: $\mathbb{N} \to \mathbb{N}$ where $f(x) = x$ is surjective, because for every $y \in \mathbb{N}$, there is an $x \in \mathbb{N}$ such that $f(x) = y$ (namely, $x = y$).
- Ex. f: $\mathbb{N} \to \mathbb{N}$ where $f(x) = 2x$ is *not* surjective, because none of the odd numbers in $\mathbb{N}$ can be mapped to (as no odd number is twice a natural number).

Bijective - A function $f(x) : A \to B$ is *bijective* if it is both injective and surjective, i.e. the function is one-to-one and every element in the co-domain is mapped to by one element in the domain.

Concept

Counting objects as a bijection between a subset of $\mathbb{N}$ and the set of those objects

Suppose you are asked to count the number of cars in a parking lot. Most likely, you will point to each car and assign it a number in your head. You point to a green car and say "1," then a black car and say "2," and so on and so forth. Now, consider that the cars in the parking lot could be called a set, say {Green car, black car, red car, 2003 Civic, Lamborghini}. Clearly, there are five elements (cars) in this set. Now, consider the subset $A = \{1, 2, 3, 4, 5\}$ of the natural numbers. In counting the cars by pointing to them and assigning them a number in a logical way, you have actually constructed a bijection from $A$ to the set of cars. See below:

$$\{\text{G, B, R, C, L}\}$$
$$\downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow$$
$$\{1,\ 2,\ 3,\ 4,\ 5\}$$

Each number (i.e. each element of $A$) is assigned to one of the cars, no number is assigned to more than one car or vice-versa, and every car is assigned a number. Therefore, counting a set of $n$ objects is equivalent to constructing a bijection with a subset $\{1, 2, 3, ..., n\}$ of the natural numbers! Later, we will use this fact when considering infinite sets.

**Focus 2** : *Every number is a cardinal number for a set of that many objects, and* $\infty = \aleph_0$ *is the cardinality of the set of natural numbers* $\mathbb{N} = \{1, 2, 3, ...\}$.

Infinity is not in the everyday sense of the word a "number." Numbers represent observable quantities, such as 3 people or $\sqrt{2}$ the length of the diagonal of a unit square. Infinity, on the other hand, is more of a concept than a number when considering the limitations of visible or observable reality. On the other hand, $\infty$ is a number in the context of cardinality. Consider that any number $n$ may be thought of as the cardinal number, or size, of a set of $n$ elements. For example, 5 can be considered the cardinal number of a set of 5 objects, be they numbers, apples, or cars. With this definition, a number is thus an abstract entity and is not tied to the specific objects themselves or what can be observed, merely how many objects there are in a set. A cardinal number $\alpha$ for a set $A$ is denoted $|A| = \alpha$. The addition of cardinals is analogous to the addition of numbers as well. If 5 and 3 are the cardinalities of sets $A$ and $B$, respectively, then $5 + 3 = |A \cup B| = 8$. The cardinality of each set added together is equal to the cardinality of the union of the two sets.
Now, when students discuss infinity, assuming they know no set theory or any of Georg Cantor's work, they are discussing the cardinality of $\mathbb{N}$, which is defined as $|\mathbb{N}| = \aleph_0$ ("aleph-naught"). We must consider three concepts before we can make sense of $\infty + \infty = \aleph_0 + \aleph_0$.

1) Cantor-Bernstein-Schröeder Theorem (denoted "CBS")
This theorem states, in short, that for 2 sets $A, B$, if there exists an injective function from $A \to B$ and another injective function from $B \to A$, then there exists a bijective function from $A \to B$. In other words, if each element of $A$ can be identified to one element of $B$ and vice-versa, then all of the elements of $A$ can be mapped injectively to all the elements of $B$; therefore, $A$ and $B$ have the same cardinality, or $|A| = |B|$.
For an example, consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. We can injectively map $A \to B$ by mapping 1 to 4, 2 to 5, and 3 to 6. Similarly, we can injectively map $B \to A$ by mapping 4 to 3, 5 to 2, and 6 to 1. Therefore by CBS, $|A| = |B|$. This is intuitively obvious, as both sets have three elements. Later, we will consider the case of larger sets in which two sets having equal cardinality (*equipollence*) is not so clear.

2) Countably infinite v. uncountably infinite.
There are two kinds of infinities - countable and uncountable. $\aleph_0 = |\mathbb{N}|$ is quite clearly countable - we achieve $\aleph_0$ by *counting* the elements of $\mathbb{N} = \{1, 2, 3, ...\}$. On the other hand, consider the set $\mathbb{R}$ of all real numbers. Intuitively, it is impossible to "count" all the real numbers. Where would one start? If one started at 0, what would one consider the second element? It is impossible to define any way of systematically counting every real number. In this way we can informally consider $\mathbb{R}$ *un*-countable. Formally, it can be shown by Cantor's ingenious diagonalization proof that it is impossible to count even the numbers of the form $a = 0.a_1a_2a_3...$, where $a_i = 0$ or $a_i = 1$ for all $i$. By CBS, this means that $|\mathbb{N}| < |A|$, where $A := \{0.a_1a_2a_3... \,|\, a_i = 0 \text{ or } a_i = 1\}$. And since $A \subset \mathbb{R}$, then clearly $\mathbb{N} < |\mathbb{R}|$.

This means that $\mathbb{R}$ is uncountably infinite, and actually gives us a definition of an uncountably infinite set:

*A set $A$ is called uncountable if $A \neq \emptyset$ and there is no surjective function from $\mathbb{N} \to A$.*

Consider that counting in itself is a way of forming a bijection between $\mathbb{N}$ and a set; when counting cars in a lot, we "assign" (or map) numbers 1, 2, 3, etc. to each car until every car is accounted for, thus forming a surjective map from the natural numbers to our set of objects. If no such map from the natural numbers to a set can be surjective - i.e. account for every object - then obviously the set in question must be uncountable. In fact, an equivalent definition of an uncountable set is a set with cardinality greater than $\aleph_0$, i.e. any set that is not countable is uncountable. (The Continuum Hypothesis, proven by Paul Cohen, actually states that it is impossible to construct a non-empty, non-finite set that is not countable nor uncountable).

3) Richard Dedekind's definition of an infinite set (Dedekind-infinite set)
The last piece of information we need to calculate $\infty + \infty = \aleph_0 + \aleph_0$ is a different definition of an infinite set. Richard Dedekind coined the definition:

*A set $A$ is infinite if there exists a proper subset $B$ of $A$ and a bijection $f$ mapping $A \to B$.*

This definition is somewhat obvious: any finite set cannot possibly bijectively map onto a *proper* subset of itself. By CBS, this would be equivalent to stating that a finite set has cardinality equal to a subset of it, e.g. that $5 = 4$, etc. However, we can consider the following example. Let $A = \mathbb{N}$ and $B = 2\mathbb{N} = \{2n \,|\, n \in \mathbb{N}\}$. Clearly, the set $B$ of positive even numbers is a proper subset of $A$, and yet the function

$$f : A \to B$$
$$n \mapsto 2n$$

is a bijective function - every natural number has one corresponding even number (its double), and no even number is not mapped to. (See below) Therefore, under this definition, $\mathbb{N}$ is infinite. More importantly, however, by CBS this also proves that the cardinality of the set of even numbers is exactly $\aleph_0$.

$$\mathbb{N} = \{1, 2, 3, 4, 5...\}$$
$$\downarrow \downarrow \downarrow \downarrow \downarrow$$
$$2\mathbb{N} = \{2, 4, 6, 8, 10, ...\}$$

Now, consider similarly that we could map $\mathbb{N}$ into the odd numbers by mapping $\mathbb{N} \ni n \mapsto (2n + 1)$, and thus that by CBS, the set of odd numbers also has cardinality $\aleph_0$.

Using these two facts and our above definition of cardinal addition as the union of two sets, we can show that

$$\begin{aligned}\aleph_0 + \aleph_0 &= |\,\{2n\,|\,n \in \mathbb{N}\}\,| + |\,\{2n+1\,|\,n \in \mathbb{N}\}\,| \\ &= |\,\{2n\,|\,n \in \mathbb{N}\} \cup \{2n+1\,|\,n \in \mathbb{N}\}\,| \\ &= |\mathbb{N}| \\ &= \aleph_0\end{aligned}$$

. Therefore, for $\infty = \aleph_0$, infinity plus infinity is indeed infinity.

**Focus 3** : *Beyond $\aleph_0$ and $\aleph_1 = |\mathbb{R}|$, there are at least countably infinitely many infinities.*

For any set $A$, one can define the *power set* of $A$ as $P(A) = \{B\,|\,B \subset A\}$. For example, if $A = \{1,2,3\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. It can be proven that $|P(A)| = 2^{|A|}$. (One can see in our example that $|A| = 3$ and $|P(A)| = 8 = 2^3$.) For finite sets, it is clear that for any set $A$, $|P(A)| = 2^n > n = |A|$. What if $|A| = \aleph_0$, though? What is $2^{\aleph_0}$?
Georg Cantor theorized that the rule that finite sets are smaller than their power sets extended to infinite sets as well. We will prove this theorem here.

**Cantor's Theorem:** *For any set $A$, $|A| < |P(A)|$.*
Since $i : A \to P(A)$ the identity function is bijective, then by CBS, it suffices to show that for *any* function $f : A \to P(A)$, $f$ cannot be surjective.
Let $f : A \to P(A)$ be a function. Consider the set $B := \{x \in B\,|\,x \notin f(x)\}$. By definition, $B \subset P(A)$. We will show that there is no $x \in A$ such that $f(x) = B$ and therefore that no mapping $f : A \to B$ is surjective.
Let $x \in A$. We have two cases.
1) $x \in f(x)$.
Then by construction, $x \notin B \Rightarrow f(x) \neq B$.
2) $x \notin f(x)$.
Then, similarly, $x \in B \Rightarrow f(x) \neq B$.
Hence $\forall\ x \in A$ and all functions $f : A \to P(A)$, $f(x) \neq B$ ($f$ cannot be surjective). Therefore, by the Cantor-Bernstein-Schröeder Theorem, $|A| < |P(A)|$.

The ramification of this theorem is that for any infinite set $A$, we can continually make larger and larger sets by taking power sets of $A$. For example, we can construct a sequence of sets each infinitely larger than the previous as such:

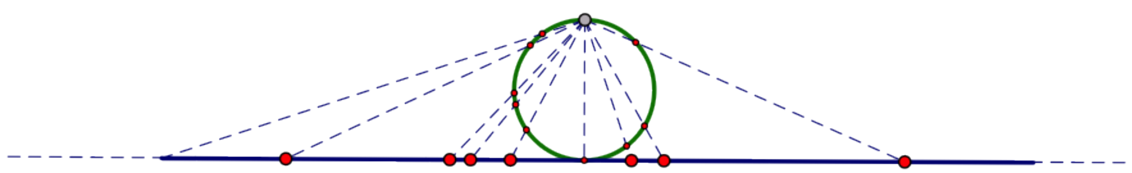
$$|\mathbb{N}| < |P(\mathbb{N})| < |P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < ...$$

We must thus be careful what we are talking about when we use the word "infinity," as there are infinitely many sets with an infinite amount of elements, and thus at least countably infinitely many cardinal numbers greater than $\aleph_0$.

**Focus 4** : *One may better understand that two sets seemingly disparate in size can have the same cardinalities if they consider a geometric representation.*

Stereographic Projection

A stereographic projection is a mapping from a sphere to a plane in any dimension. In one dimension, a sphere is a circle and a plane is a line. In this mapping, one can identify each point on a circle (of *any* size) minus one point (what one chooses as the top point of the circle) with one distinct point on the line of real numbers. For a circle with circumference 1, this informal bijection - which is formally defined below - proves that the interval (0, 1] is equipollent to $\mathbb{R}$. In fact, since this projection can be constructed with a circle with any circumference, one can prove that *any* interval $(a, b] \in \mathbb{R}$ is equipollent to $\mathbb{R}$.



Formal Definition

WLOG let our circle be $S_1 = \{(x, y) \,|\, x^2 + (y-1)^2 = 1\}$ and consider $\mathbb{R}$ as the $x$-axis in the coordinate plane.

Then the stereographic projection

$$f : (S_1 \setminus \{(0,2)\}) \to \mathbb{R}$$

$$(x_0, y_0) \mapsto (\frac{-2x_0}{y_0 - 2}, 0)$$

is a bijection.

Our points on $\mathbb{R}$ will be determined by the slope of the lines shown in the figure above; these slopes are solely determined by the coordinates of $(x, y) \in S_1$, however. Therefore, if two points on the circle are distinct, then the slopes of the lines formed in the projection will also be distinct; thus the points mapped to by $f$ are also distinct. Hence, $f$ is injective. To show $f$ is bijective is more involved, but could be shown with basic topology.

By the Cantor-Bernstein-Schrödeer Theorem, this bijection means that the cardinality of the circle, which is equivalent to the cardinality of $(0, 2\pi]$, is equal to the cardinality of $\mathbb{R}$.

**Post Commentary**:
Depending on what axioms one holds to be true, there may actually be *uncountably* infinitely many infinities. Though the set theory discussed here is not altogether germane for a grade school classroom, it is important to know the distinction between the concept of infinity as "something that is bigger than everything else" and as a rigorous mathematical idea that is indeed actually smaller than many, many other cardinal numbers. Further, it can be shown beyond addition that $\infty^\infty = \infty$ for $\aleph_0 = \infty$.

**References**

Machover, Moshe. *Set theory, logic, and their limitations.* Cambridge: Cambridge University Press, 1996.