Situation: Slopes of Perpendicular Lines

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Prompt
A teacher is discussing the slopes of lines with his/her students and tells them that, given a line with non-zero slope m, a perpendicular line will have slope $-\frac{1}{m}$. He/she then asks them to solve problems finding perpendicular lines to given lines, when a student asks, "Wait, *why* is the slope $-\frac{1}{m}$?"

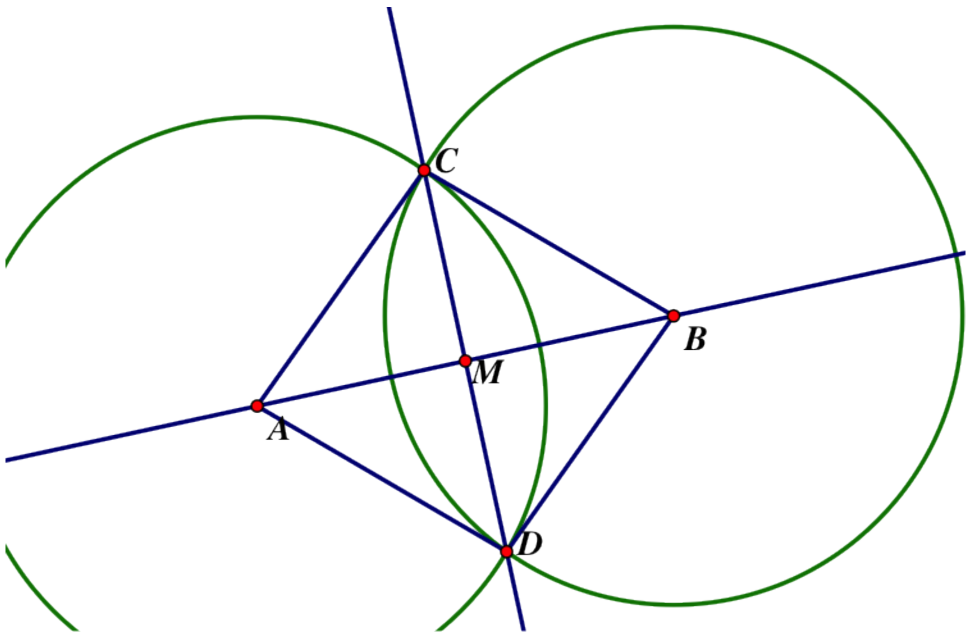
Commentary
Note: The scope of this situation is strictly Euclidean geometry. We will first discuss what it means for a line to be perpendicular to another line, prove the existence of such lines, and then show how to generate such a slope intuitively and verify it using rigid transformations, the Pythagorean theorem, similar triangles, trigonometric functions, and the dot product from linear algebra. The statement that two lines that are perpendicular have slopes that are negative reciprocals of each other is actually a theorem, and In certain foci, we will also discuss the converse of the theorem, which is also true. We will use only the term *perpendicular* in **Foci 1-4**, as opposed to the synonymous *orthogonal.* Lastly, note that m is only defined on the real numbers $\mathbb{R}$, and we will treat the cases where $m \neq 0$ or the slope is undefined in the **Post Commentary**.

Focus 1 : *Definition of perpendicular lines, proof of their existence and intuitive reasons for why* $y = mx \perp y = -\frac{1}{m}x$

Definition 10 in Book 1 of Euclid's *Elements* states: *When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is called* right, *and the straight line standing on the other is called a* perpendicular *to that on which it stands.*

Given a straight line with points A, B on it, Euclid constructed a perpendicular line as follows:
1) Extend the compass to at least one-half the length AB and draw two circles with radius AB, one with center A and one with center B.
2) Label the two intersections of the circles C & D.

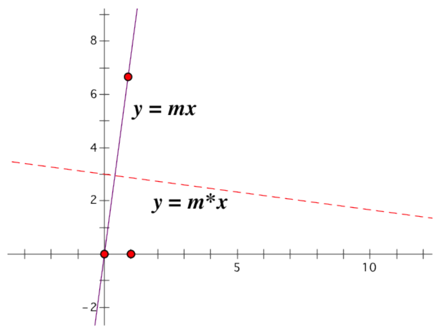
3) Draw the line through C & D. Label M the intersection with $\overline{AB}$.
Then this line is perpendicular to the line $\overline{AB}$.
Proof:
$AC = BC$, $AD = BD$, $CD = CD \Rightarrow \triangle CAD \cong \triangle CBD$
Therefore, in particular, $\angle DCA \cong \angle DCB$, $\angle CDA \cong \angle CDB$.
Also, $\angle MCA \cong \angle DCA \cong \angle DCB \cong \angle MCB$ and $\angle MDA \cong \angle CDA \cong \angle CDB \cong \angle MDB$.
By the Isosceles Triangle Theorem, $\angle CAM \cong \angle CBM$ and $\angle DAM \cong \angle DBM$.
Then by ASA, we have $\triangle CMA \cong \triangle CMB$ and $\triangle DMA \cong \triangle DMB$.
As a result, $\angle BMD \cong \angle AMD$ and $\angle BMC \cong \angle AMC$.
Each pair of angles form a straight angle, though, so we have $m\angle BMD = 90° = m\angle AMD$ and $m\angle BMC = 90° = m\angle AMC$.



This has justified the existence of a perpendicular line to any given line, and through any point (as we can choose any A, B on a line). In order to consider slope, however, one must introduce coordinates, as slope is a strictly numerical quality of a line.

Now, with coordinates and the knowledge that perpendicular lines necessarily exist given any line, let us attempt to generate the slope of such a perpendicular line intuitively.
Given any line l with slope m, we know that the line $l_2; = y = mx$ is parallel, since it has the same slope. Now, we want another line such that the intersection of the two lines forms a right angle. If, without loss of generality, $m > 0$, we would want our perpendicular to have slope m^* where $m^* < 0$. This is intuitively clear from a graph, as any two lines

with positive, non-zero slope through the origin cannot possibly be perpendicular *because the quadrant only includes ninety degrees.* See the attached *GSP* file (still working on getting this up) & picture and suppose that we have hand-drawn the red dotted line to be "approximately" perpendicular to $y = mx$.



Even as m gets larger, the perpendicular line continues to be decreasing. Thus from this picture, we know that we need slope m^* to be negative and also non-zero.
Further, we can get an idea of just how negative we want m^* to be by how large m is. In the above graph, we have a relatively large m, while m^* is not "very" negative. Rather, its slope is very slightly negative, as $y = m^*x$ is not too far from horizontal. It seems there is thus an inverse relationship of sorts between m and m^*.

Intuitively, therefore, we can reasonably guess that given a line with slope m, the slope of a perpendicular line might be $m^* = -\frac{1}{m}$, the negative multiplicative inverse of m.

Focus 2 : *The slope of a line perpendicular to a given line can be calculated using rigid transformations.*

Suppose we have the line $f_1(x) = mx + b$ and we want a line $f_2(x) = m^*x + b^*$ perpendicular to f_1. (Note: We are assuming $m \neq 0$. A line with slope 0 is horizontal, and thus a line perpendicular to it will be vertical and have an undefined slope; its equation will be of the form $x = k$, $k \in \mathbb{R}$.) Then f_1 intersects the $x - axis$ at $x = -\frac{b}{m}$, so we can construct a line perpendicular to f_1 by transforming f_1 as follows:
1) Translate f_1 by $\frac{b}{m}$ units.
2) Rotate by $+90°$ about the origin.
We will call the new line $f_2(x)$. Let t be translation and $R_{\frac{\pi}{2}}$ rotation. Then for any point $(x, f_1(x))$, the transformation gives

$$R(t(x, f_1(x))) = R_{\frac{\pi}{2}}(x + \frac{b}{m}, f_1(x))$$

$$= \left(x + \frac{b}{m} \quad f_1(x)\right) * \left(\begin{smallmatrix} cos(\frac{\pi}{2}) & sin(\frac{\pi}{2}) \\ -sin(\frac{\pi}{2}) & cos(\frac{\pi}{2}) \end{smallmatrix}\right)$$

$$= \left(x + \frac{b}{m} \quad f_1(x)\right) * \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

$$= (-f_1(x), x + \frac{b}{m})$$

We now define $f_2(x) = \{(-f_1(x), x + \frac{b}{m}) \,|\, x \in \mathbb{R}\}$. By construction, we know that $(0,0) \in f_2(x)$, so for arbitrary x we can calculate that the slope m^* of f_2 is

$$\frac{rise}{run} = \frac{(x + \frac{b}{m}) - 0}{-f_1(x) - 0}$$

$$= \frac{-(mx + b)}{mf_1(x)}$$

$$= (-\frac{1}{m})\frac{f_1(x)}{f_1(x)}$$

$$= -\frac{1}{m}$$

Therefore, by finding the line f_2 that is a 90° rotation of given f_1 and therefore perpendicular to f_1, we can calculate that the slope of f_2 is the inverse multiplicative reciprocal of the slope of f_1.

We showed here that a line rotated ninety degrees is transformed into a line with slope that is the negative reciprocal of the slope of the original line. Since transformations of the plane are invertible isometries, though, all of the above performed implications are if and only if. Therefore, the converse holds true with the same mathematical argument viewed backwards.

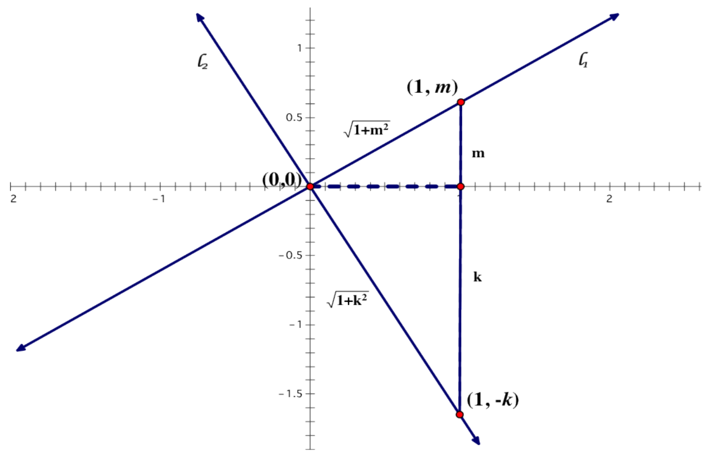
Focus 3 : *One can generate algebraically the general formula for the slope of a line perpendicular to a given line using the Pythagorean Theorem.*

Let l_1 be a line that intersects the origin ($WLOG$) with slope $m > 0$. Then $(1, m) \in l_1$. For a line l_2 perpendicular to l_1 through the origin with slope k, we know that $(1, -k) \in l_2$. ($m > 0$ implies that the angle between l_1 and the x-axis is less than 90°. Since the first quadrant contains only 90° and l_2 intersects the origin, it is necessarily the case that the y-coordinate of the point on l_2 with x-value 1 is in the second quadrant.)
Using the Pythagorean Theorem, we therefore have the following picture (see next page):

We can use the Pythagorean Theorem to determine k as follows.

$$
\begin{gathered}
(\sqrt{1+m^2})^2 + (\sqrt{1+|k|^2})^2 = (m+|k|)^2 \\
\Rightarrow 1 + m^2 + 1 + |k|^2 = m^2 + 2m|k| + |k|^2 \\
\Rightarrow 2 = 2m|k| \\
\Rightarrow |k| = \frac{1}{m}
\end{gathered}
$$

Since $k < 0$ & $m > 0$, we have that $k = -\frac{1}{m}$.



Since the converse of the Pythagorean Theorem is also true - namely, that any triple (a, b, c) such that $a^2 + b^2 = c^2$ lends itself to a right triangle with side lengths a, b & hypotenuse length c - we can easily work backwards through all of the work we have just done to prove that any two lines (of the form $y = mx + b$) whose slopes are negative reciprocals of each are perpendicular.

Focus 4 : *The right triangle argument given also has an analogue proof using similar triangles.*

Let $y = mx + b$ and $y = mx + b$ be perpendicular lines. We can translate the intersection of the two lines to the y-axis and we get a right triangle with sides in the first and second quadrant, and hypotenuse along the x-axis.
One proof of the Pythagorean Theorem involves constructing an altitude from the hypotenuse of a right triangle to the vertex of the right angle. This divides the right triangle into two smaller right triangles, which can be shown to be similar as follows (using picture

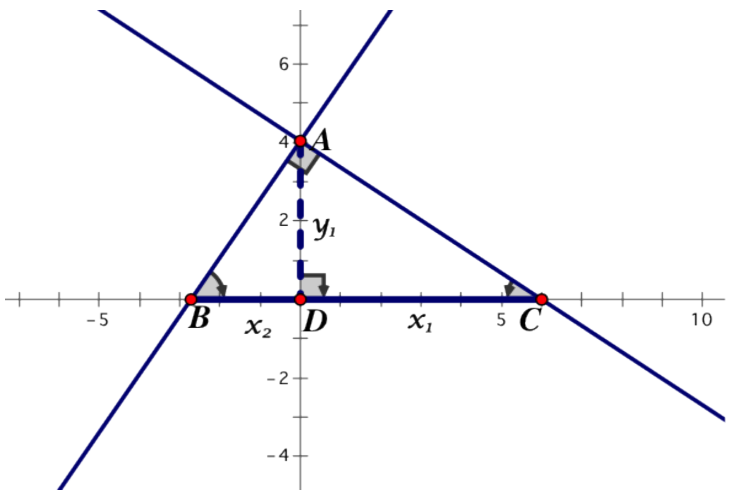
below):
$m\angle BAD + m\angle ABD = 90° = m\angle BAD + m\angle CAD \Rightarrow m\angle ABD = m\angle CAD$.
Also, $m\angle ADB = 90° = m\angle ADC$. Therefore, by AA, $\triangle ABD \sim \triangle CAD$.
Letting x_1, x_2, & y_1 represent certain side lengths, we get the proportion

$$\frac{y_1}{x_2} = \frac{x_1}{y_1}$$

But since $\frac{y_1}{x_2}$ is the slope of the line through A, B, and $-\frac{y_1}{x_1}$ is the slope of the line through A, C, we get by the previous equation that

$$\frac{y_1}{x_2} = \frac{x_1}{y_1} = -\frac{1}{(-\frac{y_1}{x_1})}$$

Therefore, the slope of $\overline{AB}$ is the negative reciprocal of the slope of $\overline{AC}$.



Focus 5 : *The slope of a non-horizontal or -vertical line and the slope of a line perpendicular to it may be evaluated using basic trigonometric functions.*

Let $f(x) : ax + by = c$ be a line that meets the x axis at angle θ. Then the slope of f may be expressed by $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$.

Hence, any line tangent to f will have a slope of

$$\tan(\theta + \frac{\pi}{2}) = \frac{\sin(\theta + \frac{\pi}{2})}{\cos(\theta + \frac{\pi}{2})}$$

$$= \frac{\sin(\theta)\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})\cos(\theta)}{\cos(\theta)\cos(\frac{\pi}{2}) - \sin(\theta)\sin(\frac{\pi}{2})}$$

$$= \frac{\cos(\theta)}{-\sin(\theta)}$$

$$= -\cot(\theta)$$

$$= -\frac{1}{\tan(\theta)}$$

Thus, the slopes of perpendicular lines are negative reciprocals of each other.

Focus 6 : *The slope of a line perpendicular to a given line can also be calculated, or confirmed, using the inner product of linear algebra.*

In linear algebra, the inner product, or dot product, of two vectors $\vec{u} = (u_1, ...u_n)$, $\vec{v} = (v_1, ...v_n)$ for $n \in \mathbb{N}$ is defined geometrically as

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| cos(\theta)$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.
Consider an arbitrary line $y = mx + b$. We will choose two on points are $y = mx + b$ to calculate the slope of the line, specifically $(0, b)$ and $(-\frac{b}{m}, 0)$. As done in **Foci 2**, we can translate $y = mx + b$ such that the two chosen points become $(\frac{b}{m}, b)$ and $(0, 0)$. Then the vector $\vec{v} = (\frac{b}{m} \ b)$ will have the same slope (as a result of its direction and magnitude) as $y = mx + b$. Consider any vector $\vec{u}$ such that $\vec{u} \perp \vec{v}$ and $\vec{u}$ intersects the origin. Then we can define the vector $\vec{u}_{x,y}$ as the vector starting at $(0, 0)$ and ending at (x, y), for *any* $(x, y) \in \vec{u}$. Then we have

$$\vec{u}_{x,y} \perp \vec{v}$$

$$\Rightarrow \vec{u}_{x,y} \cdot \vec{v} = \|\vec{u}_{x,y}\| \|\vec{v}\| cos(90°)$$

$$\Rightarrow (x \ y) \cdot (\frac{b}{m} \ b) = 0$$

$$\Rightarrow x * \frac{b}{m} + b * y = 0$$

$$\Rightarrow x * \frac{\frac{b}{m}}{-b} = y$$

$$\Rightarrow y = (-\frac{1}{m}) * x$$

Therefore, for *any* (x, y) on $\vec{u}$ orthogonal to $\vec{v}$, we have that y is the negative reciprocal of the slope of $\vec{v}$ times x, and therefore that the slope of $\vec{u}$ is itself the negative reciprocal of m. Since $\vec{v}$ was just a vector representation of $y = mx + b$, the orthogonality of $\vec{u}$ and $\vec{v}$ is synonymous to the perpendicularity of $y = mx + b$ and the $y = -\frac{1}{m}x + b^*$.

One of the properties of the inner product is that any two vectors whose inner product is zero are perpendicular. To prove the converse of our theorem, then, consider two vectors $\vec{u}, \vec{v}$ through the origin with slopes m and $-\frac{1}{m}$. Then $(1, m) \in \vec{u}$ & $(1, -\frac{1}{m}) \in \vec{v}$ and

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (1, m) \cdot (1, -\frac{1}{m}) \\ &= 1 - 1 = 0 \end{aligned}$$

Hence, $\vec{u} \perp \vec{v}$.

Post Commentary

We did not yet consider the cases when $m = 0$ or a line has undefined slope.
If $m = 0$, then $y = mx + b = b$ is horizontal, and any line perpendicular to $y = b$ must have undefined slope. i.e. be vertical. One can show this by considering the limit of the slopes of lines perpendicular to $y = mx + b$ as $m \to 0$.

$$\lim_{m \to 0} -\frac{1}{m} = \infty$$

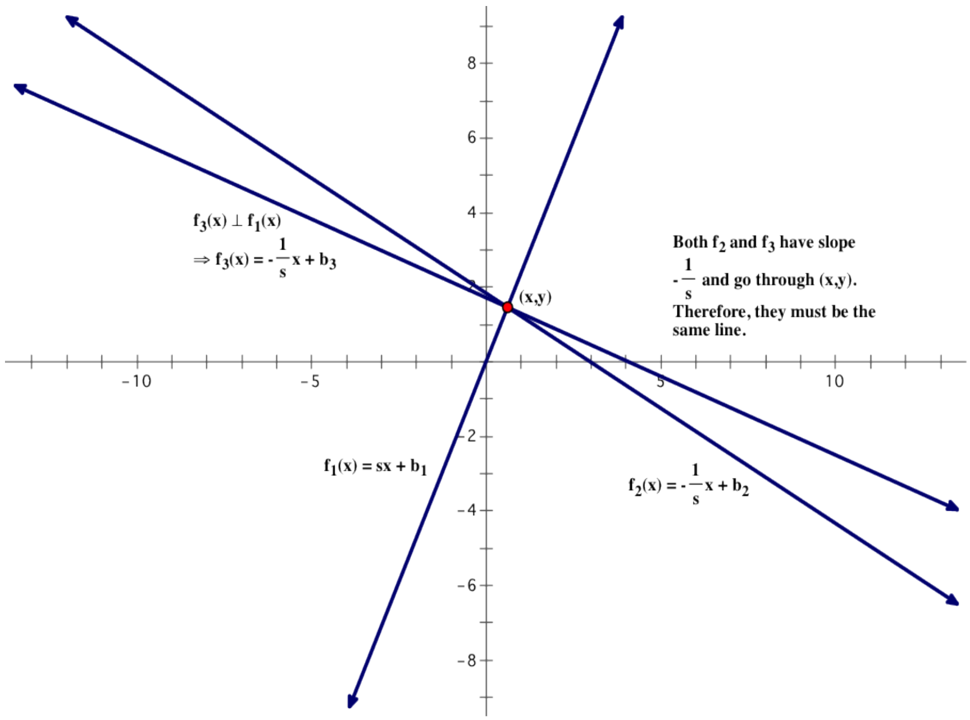
As $m \to 0$, $-\frac{1}{m}$ approaches ∞ and $y = -\frac{1}{m}x + b$ approaches a vertical line.
Similarly, if the slope of a line is undefined and has the form $x = k$, we can think of all lines perpendicular to those of the form $x = my + k(\equiv \frac{1}{m}x - \frac{k}{m} = y$ as $m \to 0$.

$$\lim_{m \to 0} -\frac{1}{\frac{1}{m}} = \lim_{m \to 0} m = 0$$

As $m \to 0$, $x = my + k$ approaches the vertical line $x = k$, and the line $y = -mx - \frac{k}{m}$ perpendicular to it approaches a horizontal line.

A clever proof of the converse of the theorem given in the prompt

Suppose $f_1(x)$ & $f_2(x)$ are two lines whose slopes are negative reciprocals of each other, say $s :=$ slope$(f_1(x))$, $-\frac{1}{s} =$ slope$(f_1(x))$. Then we necessarily know that $f_1 \not\parallel f_2$ and hence that f_1 intersects f_2 at some point (x, y). Now, suppose that $f_1(x) \not\perp f_2(x)$. By the theorem we have proved multiple ways in this Situation, we can construct a new line f_3 perpendicular to f_1 that intersects f_1 at (x, y) and necessarily has slope $-\frac{1}{s}$. But then f_3 and f_2 are two lines with the same slope that go through the same point. Since any line is defined by its slope and a point, we have that $f_3 \equiv f_2$, which contradicts our assumption that $f_3 \not\equiv f_2$. Therefore, $f_2 \perp f_1$. (Picture below.)



Determining the slopes of lines perpendicular to given lines therefore largely makes use of generating numbers using algebraic formulas. Here, we have at the least tried to present some form of intuitive reasoning that could shine light on what many students understand as a formulaic entity, as well as having provided calculations from ranging sub-disciplines of mathematics inherently separate from algebra in hopes of adding more substance to the subject of perpendicular lines.

Resources

http://aleph0.clarku.edu/ djoyce/java/elements/bookI/bookI.html.