Situation – Zero v. Nothing

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Prompt

A man's four-year old asks him, "What is the difference between 0 and nothing?" and the man is completely tongue-tied.

Commentary

In everyday conversation, zero and nothing may be used synonymously, ad reasonably so. Mathematically, though, the differences between zero and nothing are well defined, and can be observed in different contexts. We will consider 0 as a numeral, a number, a probability, a measurement, and as the set $\emptyset = \{\}$, and contrast this with what "nothing" means in each of these contexts.

Mathematical Foci

Mathematical Focus 1

A working definition of 'nothing'

While used without reservation in everyday language, "nothing" as a linguistic entity is paradoxical and somewhat self-defeating; it is something, a word, representing the lack of something. When one says they have "nothing on their mind," they mean that there is a *lack* of anything on their mind. This paradox stumped many an ancient mathematician, and actually led to much historical delay in the acceptance and use of zero as a numeral representing nothing, let alone being used in calculations. While we define zero in every other foci and, in **Foci 4** present a different definition of nothing, we will here define nothing as we discuss it in **Foci 2, 3, 5, & 6**. Let us note and appreciate that even our definition of nothing is an act of attributing it certain properties; nonetheless, our paradoxical definition will suffice.

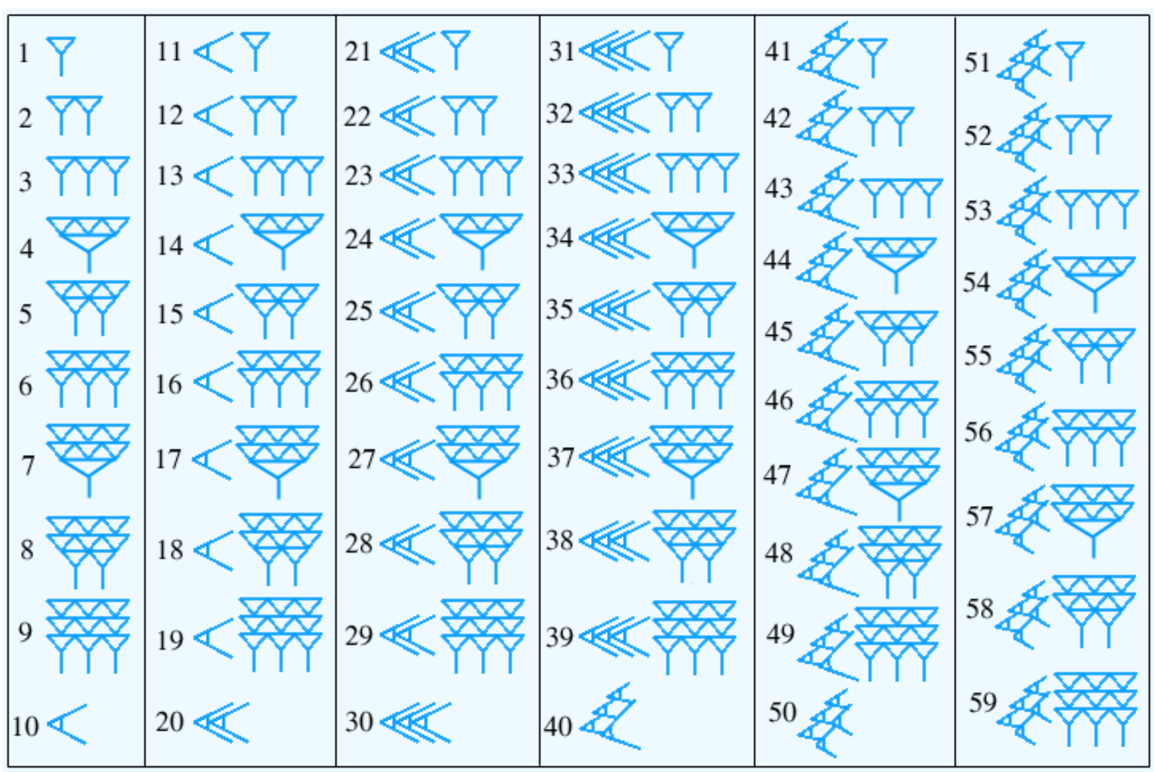
Nothing is literally no thing. It is the absence of any object or thing, and thus has no properties besides *the property of having no properties. Nothing can therefore be said to have only this one vacuous property.*

Mathematical Focus 2

Historically, 0 was used as a numeric placeholder in ancient civilizations in Babylon, Greece, Egypt, and India. It was a representation of nothing, and therefore distinct from nothing.

Though the debate about the origins of zero as a numerical placeholder are hotly debated (as many cultures have much pride at stake in the matter), it is generally accepted that the mathematicians and astronomers of ancient Babylon were the first to use a symbol representing the lack of any number. Though not the same as our ovoid 0, the concept is the crucial part of the mathematics.

Babylonian mathematicians as early as 1830 BCE used a sexagesimal (base 60) numerical system of counting and had symbols for the numbers 1-9. They wrote their numbers in soft red clay using a wedge and a stylus. Below is a table of their numerals for 1-59:



We will call the numeral for 1 a “wedge,” and for 10 a “hook.” In doing so, note that the Babylonians incorporated base 10 into their sexagesimal system, most likely due to the number of fingers on the human hand. Addition in the Babylonian system allowed for “carrying,” in which one hook and seven wedges plus three more wedges would produce one hook and ten wedges, or two hooks.

For the number 60, Babylonians would simply draw a *large* wedge, representing 1*60. Therefore, the number 132 = 2*60 + 12 was represented by two large wedges, one hook, and two small wedges. Problems arose in this system, however, when considering, for example, the numeral for 3609 = 1*3600 + 9. This would necessarily involve ten wedges – one for multiples of $60^2 = 3600$, and nine for a multiple of 1. One could theoretically draw larger and larger wedges to represent larger powers of 60, but this process begged for mistakes to be made and wedges to end up looking alike. How, then, could one distinguish the ten wedges representing 3609 and ten wedges representing, say 69 = 1*60 + 9? The solution was a symbol for nothing, or what we now know as 0. That symbol is shown here:

This symbol represented none of a certain digit. Therefore 3609 would become 3609 = 1*3600 + 0*60 + 9*1, or (from left to right) a wedge, two sideways wedges (the symbol for nothing), and nine wedges.

It is important to note that this symbol, a numeral, serves the same purpose as our 0 in the number 307 = 3*100 + 0*10 + 7*1. On the other hand, however, the Babylonian double sideways wedge also was never used at the end of a number, in decimals, nor on its own.

The zero the Babylonians used as a placeholder would lead to other symbols and, with them, uses for the placeholder. The Greeks, who captured the Babylonian empire in 331 BCE, would change the sideways double wedge into the little circle ° (which we now know as the symbol for degrees) that more closely resembles our 0, and would also utilize the placeholder within their base 10 system in *decimals*. (Note: The Greeks assigned each of their 24 letters, as well as 3 other symbols, to each of the numbers $k * 10^l$, where $k \in \{1, 2, \dots, 9\}$ and $l \in \{0,1,2\}$.) For one specific example, consider that of the astronomer Ptolemy. Around 150 AD, Ptolemy in his magnum opus *Almagest* would write

$$\overline{\mu\alpha\ o\ \iota\eta}$$

which represented $41°00'18''$. The $\bar{o}$ represented *zero minutes* in Ptolemy's trigonometry (which we will not detail here).

Though owing largely to the Greek and Babylonian traditions, zero as a placeholder used in either the middle, end, or decimal position of numbers was not fully utilized until the later half of the first millennium in India with mathematicians such as Brahmagupta, Mahavira, and Bhaskara.

This review of history, though brief, illustrates that zero, or 0 as we know it, is a numeric placeholder, and helps us distinguish between numbers like 31 and 301, 0.1 and 0.001, etc. Nothing, on the other hand, is exactly nothing, let alone a mathematical representation of itself.

Mathematical Focus 3

0 is a number, and is the additive identity element of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R},$ *and* $\mathbb{C}$*. It can be added, subtracted, or multiplied by other numbers. Nothing, by definition, has no such properties.*

Among the most easily noticeable differences between zero and nothing is actually semantic: zero can describe something, whereas "nothing" cannot. For example, if there are 2 cars in a driveway, and then both cars leave, there are indeed 0 cars left in the driveway. Zero in this context describes an amount of objects. Nothing, on the other hand, makes no sense here. There cannot be "nothing" of "something" semantically, and the line is similarly drawn here between nothing and zero in mathematics.

Mathematical Focus 4

Nothing and zero have a different meaning in probability as well. Any event that is impossible (i.e. has no probability of happening) has a *probability of zero; on the other hand, an event with zero probability is not necessarily impossible.*

In the context of probability, we will define "nothing" as *impossible*. Nothing implies the lack of anything, hence the *lack of probability* is the absence of any probability – even a probability of 0.

Now, the difference between impossibility and "zero probability" is not negligible. Suppose we have a rectangular dartboard with all the numbers in [0, 1] on it – irrational and rational. Consider:

1) What is the probability that we will hit $\sqrt{-1} = i$ on the dartboard?
2) What is the probability that we hit $\frac{3}{5}$?

For question (1), the answer is clear: the probability that we hit i on the dartboard is zero because it is *impossible*. There is *no probability* that we will hit an imaginary number on our real number dartboard, and thus there is zero probability of hitting i.
On the other hand, what is the probability that we hit $\frac{3}{5}$? Certainly it is not impossible; this fraction is on our dartboard, so there is absolutely a chance that we hit it. However, because the interval [0,1] is uncountably infinite and $\frac{3}{5}$ as a point is infinitesimally small, the probability of hitting it with a dart is indeed zero.
Therefore, zero and nothing are not equivalent in the context of probability, for while impossibility necessarily implies zero probability, the converse does not hold, as in the example given above.

Mathematical Focus 5

0 as a measure; nothing as, well, nothing.

Mathematical Focus 6

In set theory, nothing is literally nothing and has no definition, whereas 0 is defined as 0 = $\{\} = \emptyset$*, the set containing nothing, or the empty set. Nothing is thus contained in 0.*

In Zermelo-Frankel axiomatic set theory, a set must be "something." In the construction of natural numbers as ordinals (certain kinds of sets that we will not define here; for a full discussion see Machover), 0 is the empty set, or null set. Note that the empty set is, nonetheless, a set; it is vacuously a set, because it contains no objects or is a set categorized by what it does *not* contain. On the other hand, nothing is not a set or an object in and of itself, but is literally nothing and is vacuously contained in 0. Again, as in **Focus 1**, 0 is in set theory a set with properties – namely, that it contains nothing and is contained in every other *non*-empty set – whereas nothing is not an entity and has no properties.

Though the difference between 0 and nothing is here again paradoxical in that a set can "contain" nothing, we rely on the axiomatic notion that a set – even one with nothing in it – is still *something,* a proper set, with properties and a rigid definition.

Post Commentary

Analyzing the differences between zero and nothing may actually lead to a rather worthwhile consideration of the philosophical relationship between mathematics and language, which has been a source of inspiration for such mathematicians and philosophers as Ludwig Wittgenstein and Kurt Gödel.

In **Focus 4**, we utilized strictly *theoretical* probability. One could also consider experimental probability, in which impossibility and zero probability are still not equivalent. Consider rolling an ordinary six-sided die. Suppose that five rolls result in 1,3,3,5,1, & 3. Then the *experimental probability* – the ratio of desired outcomes to overall trials – of rolling an even number is 0/6 = 0. Similarly, the experimental probability of rolling a 7 is 0, as rolling a 7 on a traditional six-sided die is impossible. However, if a seventh roll results in a 2, 4, or 6, then the experimental probability of rolling an even number becomes $\frac{1}{7} \approx .143$, while the experimental probability of rolling a 7 will remain 0 for any possible amount of trials.

Resources

Machover, Moshe. *Set theory, logic, and their limitations.* Cambridge: Cambridge University Press, 1996.

“Difference Between Zero and Nothing.”
http://mathforum.org/library/drmath/view/52387.html

http://easycalculation.com/funny/numerals/babylonian.php