Situation: Complex Roots in Conjugate Pairs

Prepared at University of Georgia Center for Proficiency in Teaching Mathematics
June 30, 2013 – Sarah Major

Prompt:

A teacher in a high school Algebra class has just explained all of the methods of solving quadratic equations and discussed that some polynomials may produce complex solutions. These solutions should be written in the form $a \pm bi$. A student then asks, "Do all complex solutions occur in conjugate pairs like this?"

Commentary:

The concept of finding roots for polynomial equations stems from the Fundamental Theorem of Algebra proven by Carl Friedrich Gauss. Two important properties come from this theorem. First of all, it states that every single-variable polynomial that is not constant and contains complex coefficients has at least one complex root (though they may be real because every real number can be written as a complex number, which will be shown later). Secondly, every polynomial with the same properties of degree n has exactly n roots (though one or more of these roots may have multiplicity) within the complex numbers. Teachers should be familiar with this theorem and how it can be proven a number of ways. However, providing a proof in this Situation would take away from the actual prompt and the focus on complex conjugate pairs.

As a technicality, when we speak of complex conjugate pairs, we are referring to non-real numbers. We could say that these non-real numbers are all numbers that can be written is the form $a \pm bi$, and this set would still contain all real numbers since the conjugate of a real number x written in the complex form $x + 0i$ is $x - 0i$, and therefore, the complex conjugate of every real number written in standard complex form is itself. However, this statement can be seen intuitively and actually comes into play in the proof of the Complex Conjugate Root Theorem proven later.

Mathematical Foci:

Mathematical Focus 1
Complex numbers adhere to certain arithmetic properties for which they and their complex conjugates are defined.

Every complex number can be written in the standard form $a + bi$ where a and b are both real. Also, a is known as the *real* part and b is known as the imaginary part because it is multiplied by the imaginary number i. Other rules for complex numbers are as follows:

1. $i^2 = -1$
2. $a + bi = 0 \leftrightarrow a = b = 0$
3. $(a + bi) + (c + di) = (a + c) + (b + d)i$
4. $(a + bi)(c + di) = (ac - bd) + (bc + ad)i$

Every complex number written in standard form has a complex conjugate, which is simply taking the imaginary portion and inverting the sign. Since the standard form of a complex number is $a + bi$, the complex conjugate is always written in the form $a - bi$. The notation for a complex conjugate usually involves putting a bar over the variable being used. For example, if $z = a + bi$, then $\bar{z} = a - bi$. One particular property that these pairs hold is that their product is always a non-negative real number. The following proves this statement:
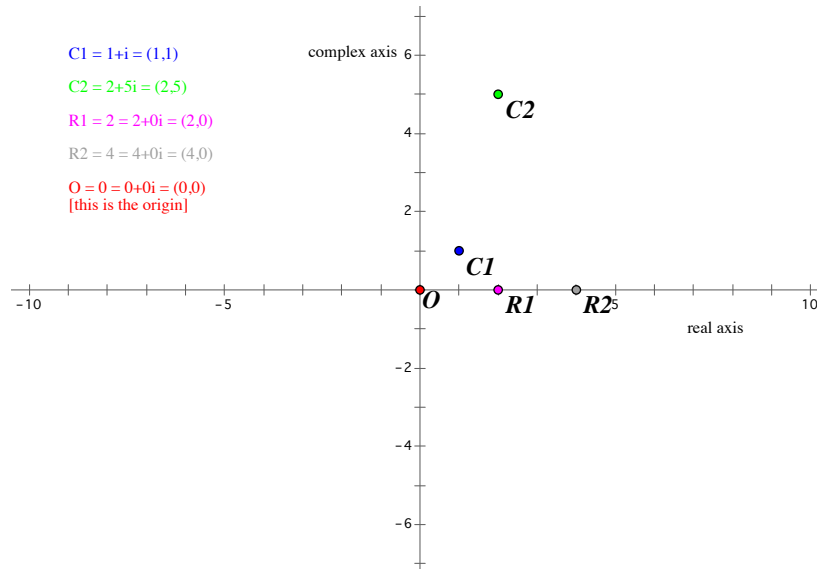
$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2i^2 = a^2 + b^2$$

This fact is used in the division of complex numbers to remove any imaginary portions from the denominator (often called *rationalization* or *rationalizing the denominator*).

Mathematical Focus 2

Complex numbers can be represented as points and/or vectors in the complex plane. This can then be related to the Cartesian plane when dealing with graphs of complex roots.
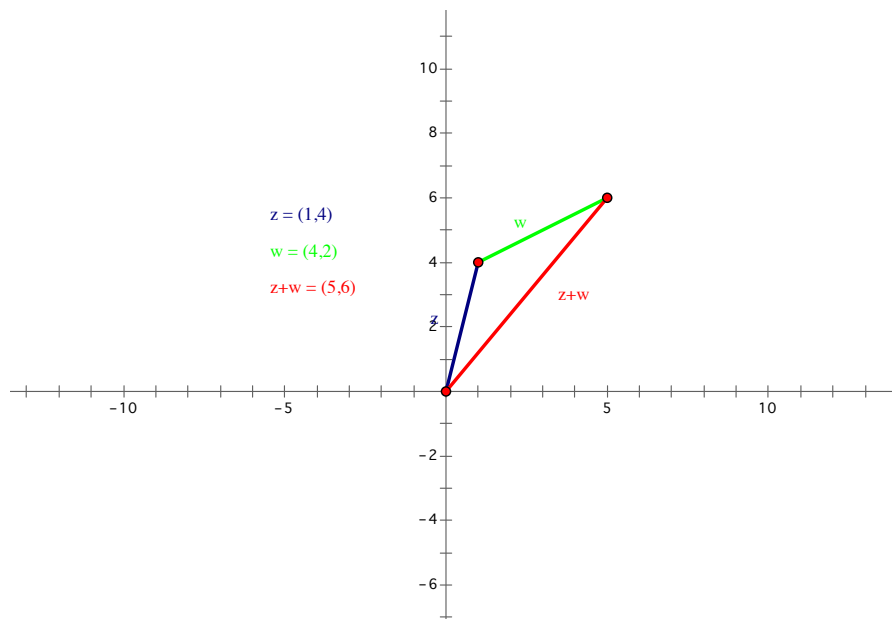
As stated in **Focus 1**, $z = a + bi$ is an imaginary number where a is the real part and b is the imaginary part. We can therefore plot z as a point (a, b) in the complex plane where the horizontal axis is the "real axis" and the vertical axis is the "imaginary axis". Intuitively, we can see that any real number x can be written as a point $(x, 0)$ on the complex plane, and the plot of any real point will lie only on the real axis. This also shows that real numbers are a subset of complex numbers.

C1 = 1+i = (1,1)

C2 = 2+5i = (2,5)

R1 = 2 = 2+0i = (2,0)

R2 = 4 = 4+0i = (4,0)

O = 0 = 0+0i = (0,0)
[this is the origin]

complex axis

C2

C1

O   R1   R2

real axis

This also means that complex numbers can be represented as vectors. For instance, the point $z = a + bi$, z can be represented as the vector (a, b). In this way, we can visualize addition of complex numbers as vector addition in a two-dimensional plane. For example, **Focus 1** explains the arithmetic rules of complex numbers, so given two complex numbers, $z = 1 + 4i$ and $w = 4 + 2i$, we know that $z + w = (1 + 4) + (4 + 2)i = 5 + 6i$. However, this operation can be done through simple vector addition:
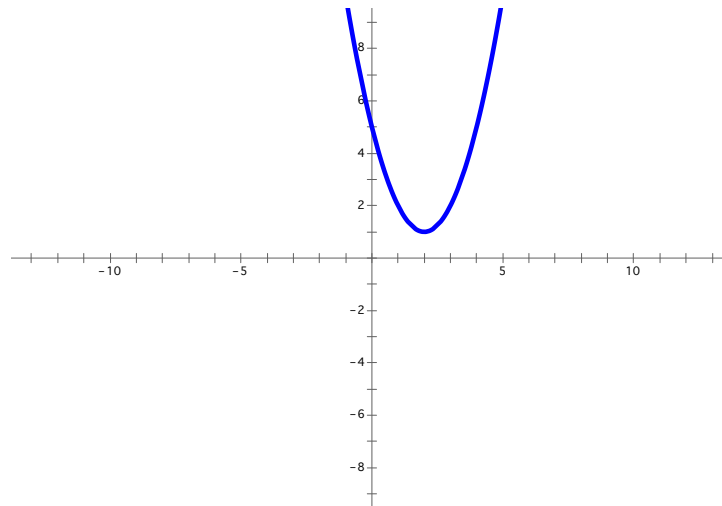
$$z + w = (1,4) + (4,2) = (1 + 4, 4 + 2) = (5,6) = 5 + 6i$$

We can also perform this operation graphically by plotting one vector and then lining up the second vector on the end of the first.

$z = (1,4)$
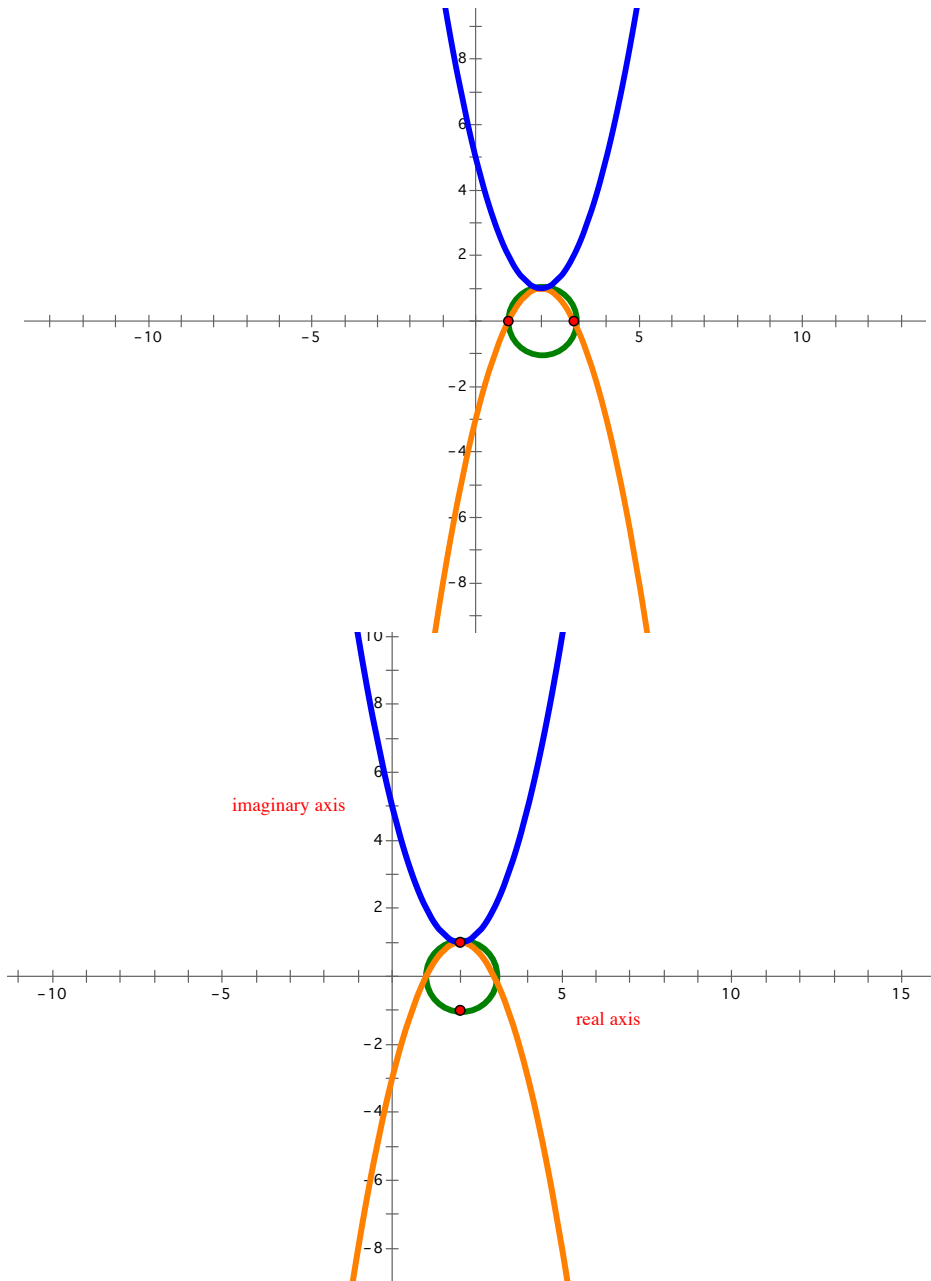
$w = (4,2)$

$z+w = (5,6)$

w

z+w

Graphing the representation of two complex numbers being subtracted is similar, except the second vector goes in the opposite direction.

There is a graphical manipulation that can be used to find complex roots of quadratic equations. Take the quadratic equation $f(x) = x^2 - 4x + 5$. If we use the quadratic formula, we find that the two roots occur at $x = 2 \pm i$. However, when we look at the graph, we are not able to perceive where these roots occur because the graph never crosses the x-axis, and this is the stipulation for being a root.

However, there is a trick to finding these roots graphically. We simply flip the parabola at its vertex, find the roots of the flipped parabola, and mark these two points. Create a circle that passes through these two points and then rotate the circle 90°. The new locations of the two points display the values of the complex roots if the graph appeared in the complex plane.

As we can see from the second graph, the roots would be located at $(2,1)$ and $(2,-1)$, or $2+i$ and $2-i$. This method works due to the location of the complex plane in reference to the Cartesian plane. We can think of the Cartesian plane as the wall of a room, and the complex plane is the floor. Both planes coexist together as functions are graphed, and this is why some equations may have both real and imaginary roots. We can only see the Cartesian plane in a two-dimensional graph, but the function actually occurs three-dimensionally. While any transformation left, right, up, or down still produce real points, movements backwards or forwards denote movement along the imaginary axis and therefore imaginary points.

Mathematical Focus 3

The Complex Conjugate Root Theorem states that complex roots always appear in conjugate pairs.

The Complex Conjugate Root Theorem is as follows:

Let $f(x)$ be a polynomial with real coefficients. Suppose $a + bi$ is a root of the equation $f(x) = 0$, where a and b are real and $b \neq 0$. Then $a - bi$ is also a root of the equation.

A simple proof of this theorem is as follows:

We have the polynomial $f(x)$ such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

where all a_r are real. Then

$$\overline{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n} = \overline{0}$$
$$\overline{a_0} + \overline{a_1 x} + \overline{a_2 x^2} + \cdots + \overline{a_n x^n} = 0$$
$$\overline{a_0} + \overline{a_1}\overline{x} + \overline{a_2}\overline{x}^2 + \cdots + \overline{a_n}\overline{x}^n = 0$$
$$a_0 + a_1 \overline{x} + a_2 \overline{x}^2 + \cdots + a_n \overline{x}^n = 0$$

As stated in the **Commentary**, all real numbers are their own complex conjugates, making the above statement true. Therefore,

$$f(\overline{x}) = 0$$

The previous statement shows that when a complex root x satisfies the equation $f(x) = 0$, its complex conjugate, $\overline{x}$, also satisfies the equation, so these pairs always occur together.

Mathematical Focus 4

If the degree of a polynomial is even and contains complex solutions, then the complex solutions will appear in conjugate pairs.

This can be easily shown through the quadratic formula, which is as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We know that the discriminant of a quadratic function determines the amount and types of roots the polynomial has. The properties are as follows:

1. $b^2 - 4ac = 0$, there is one real root that repeats.
2. $b^2 - 4ac > 0$, there are two distinct real roots.
3. $b^2 - 4ac < 0$, there are two distinct complex roots that are conjugates.

This is because taking the square root of the discriminant produces a complex number, and due to the $\pm$ that appears before the square root, conjugates are created. However, the quadratic formula technically only applies to quadratic polynomials, but polynomials of a higher even degree can be manipulated so that this formula can be used. For instance, given the biquadratic formula

$$ax^4 + bx^2 + c = 0$$

we can set $z = x^2$. After substituting, we have

$$az^2 + bz + c = 0$$

which is the form necessary to use the quadratic formula. However, the standard form of a quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

cannot use this same method. Therefore, we will have to use the discriminant of a quartic equation to figure out how many and what types of solutions will occur for this equation. The discriminant is:

$$256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2$$

The cases for this discriminant are as follows:

1. $\Delta = 0$, then the root has multiplicity or is the square of a quadratic (see situation above). In the event that there are complex roots, they appear in conjugate pairs.
2. $\Delta > 0$, then the four roots are either all real distinct roots or are two pairs of complex conjugate roots.
3. $\Delta < 0$, then there are two real roots and two complex conjugate roots.

Mathematical Focus 5

If the degree of a polynomial is three and contains complex solutions, then the complex solutions will appear in conjugate pairs.

The general standard form of a cubic equation is

$$ax^3 + bx^2 + cx + d = 0$$

The number and nature of the roots can be found through evaluating the discriminant:

$$18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2$$

1. $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 = 0$, there are three real roots and at least two are equal.
2. $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 > 0$, there are three distinct real roots.
3. $18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 < 0$, there is one real root and two complex roots that are conjugates.

Mathematical Focus 6

For polynomials of degree five or higher, there is not a general algebraic method of finding the solutions. However, this does not mean that there are not any solutions.

The Abel-Ruffini theorem says that unlike polynomials of degree four or lower, there is no general algebraic solution involving radicals for polynomials of degree five or higher. However, this does not mean there are not any solutions to such polynomials. According to the Fundamental Theorem of Algebra, every polynomial that is not a constant has at least one complex solution (though this solution may be real). Also, some fifth degree or higher polynomials can be factored into solvable factors (for instance, a quadratic times a linear function).

For the fifth degree or higher equations that cannot be factored, one method of finding approximations of roots would be Newton's method (also known as the Newton-Raphson method). In this method, one begins with an initial guess, x_0, and uses the derivative of the function value and the actual function value to find the next approximation. This process is continued until a sufficiently accurate approximation is reached. The process looks like this:

$$x_0 = x_0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\vdots$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Lesser iterations of the methods can be used if the initial guess is closer to the actual value of the root.

Another method of finding approximations for roots is the Laguerre method, which is more likely to converge at an actual root value given any initial guess. Similar to Newton's method, one begins with an initial guess, x_0, but a different process is used. $x_{n+1} = x_n - a$, where

$$a = \frac{n}{\max\left[G \pm \sqrt{(n-1)(nH - G^2)}\right]}$$

$$H = G^2 - \frac{f''(x_n)}{f(x_n)}$$

$$G = \frac{f'(x_n)}{f(x_n)}$$

The process is repeated until a becomes small enough or the maximum number of iterations has been reached.