Situation #2: False Identity

Prepared at University of Georgia
William Plummer
EMAT 6500
Date last revised: July 28, 2013

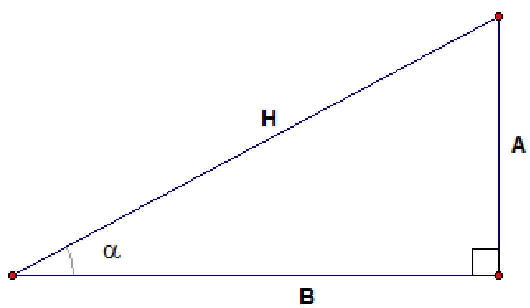
Prompt

A Trigonometry class is deriving the Pythagorean Identities from the unit circle and arrives at the following identity: $\sin^2(x) = 1 - \cos^2(x)$. One of the students decides that the following must also be an identity:

$$\sin(x) = \sqrt{1 - \cos^2(x)}$$

Mathematical Focus 1

The values of trigonometric functions of an angle α can be determined by evaluating the ratios of the lengths of the sides of a right triangle that includes an internal angle α.



There are 6 possible ratios that can be evaluated in each right triangle which are defined as follows:

sine (α) = $\frac{A}{H}$

cosecant (α) = $\frac{1}{\text{sine}(\alpha)} = \frac{H}{A}$

cosine (α) = $\frac{B}{H}$

secant (α) = $\frac{1}{\text{cosine}(\alpha)} = \frac{H}{B}$

tangent (α) = $\frac{\text{sine}(\alpha)}{\text{cosine}(\alpha)} = \frac{A}{B}$

cotangent (α) = $\frac{1}{\text{tangent}(\alpha)} = \frac{\text{cosine}(\alpha)}{\text{sine}(\alpha)} = \frac{B}{A}$

Similarly, the 6 trigonometric ratios defined above can also be derived from the unit circle, a circle with center (0,0) and a radius of 1 unit. A unique point F on the unit circle can be associated with any real number α in the following manner:

- When α = 0, F is (1,0)
- When α is positive, F is found by proceeding a distance |α| in the counterclockwise direction from the point (1,0).
- When α is negative, F is found by proceeding a distance |α| in the clockwise direction from the point (1,0).

If α is a real number and F(x,y) is the point on the unit circle, then the 6 trigonometric functions of α are defined as follows:

sine (α) = y

cosecant (α) = $\frac{1}{y}$ (if y ≠ 0)

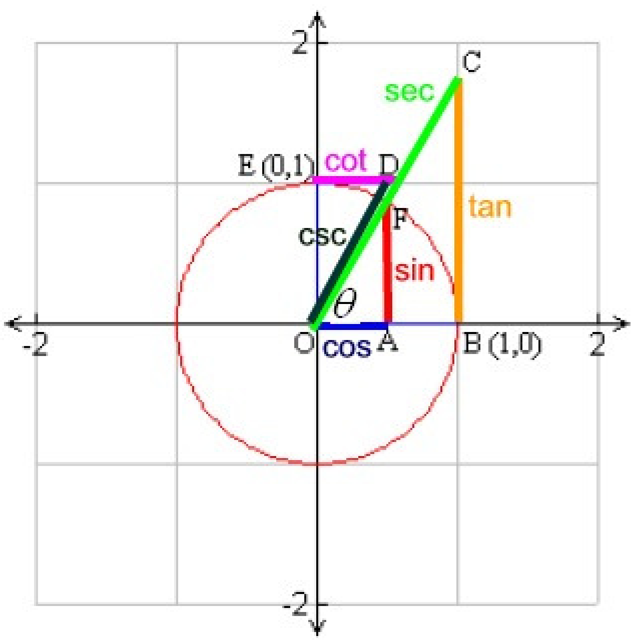
cosine (α) = x

secant (α) = $\frac{1}{x}$ (if x ≠ 0)

tangent (α) = $\frac{y}{x}$ (if x ≠ 0)

cotangent (α) = $\frac{x}{y}$ (if y ≠ 0)

All 6 trigonometric functions can be graphed on the unit circle as follows:



Mathematical Focus 2

The Pythagorean Theorem is used to derive Trigonometric Identities which are obtained from the Trigonometric functions determined from right triangles formed by points on the Unit Circle.

From the unit circle above, it can be seen that any value of α and its associated point on the circle create a right triangle with sine (α) and cosine (α) as the legs and the radius as the hypotenuse. From the Pythagorean Theorem, we know the following:

$$\sin^2(\alpha) + \cos^2(\alpha) = 1$$

By subtracting either $\cos^2(\alpha)$ or $\sin^2(\alpha)$ from both sides, we have:

$$\sin^2(\alpha) = 1 - \cos^2(\alpha)$$

and

$$\cos^2(\alpha) = 1 - \sin^2(\alpha)$$

By taking these three equations and dividing both sides by either $\cos^2(\alpha)$ or $\sin^2(\alpha)$ we obtain the remaining 6 Pythagorean Trigonometric Identities.

$$\tan^2(\alpha) + 1 = \sec^2(\alpha) \qquad 1 + \cot^2(\alpha) = \csc^2(\alpha)$$

$$\tan^2(\alpha) = \sec^2(\alpha) - 1 \qquad \cot^2(\alpha) = \csc^2(\alpha) - 1$$

$$1 = \sec^2(\alpha) - \tan^2(\alpha) \qquad 1 = \csc^2(\alpha) - \cot^2(\alpha)$$

Mathematical Focus 3

An identity is a statement that two quantities are equal which is true for all values of the variables for which the statement is meaningful. For example:

(a) $x + 3 = 3 + x$ is an identity since it is true for all values of x.

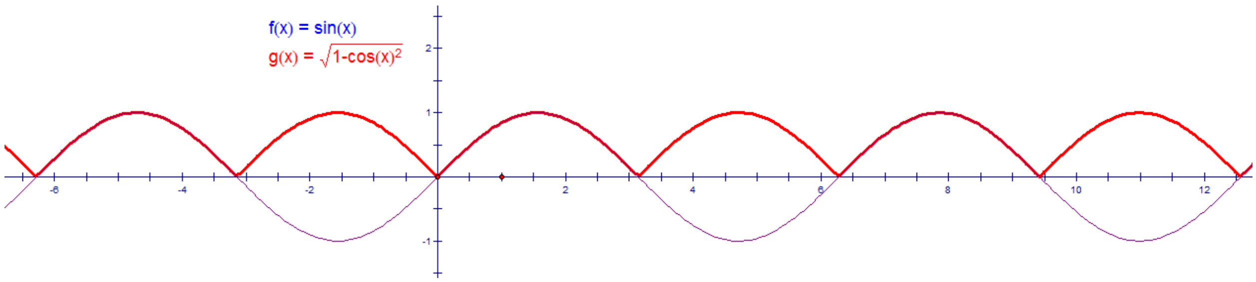
(b) $x + 3 = 5$ is not an identity because it is only true for $x = 2$

(c) $x \cdot \frac{1}{x} = 1$ is an identity since it is always true unless $x = 0$, in which case it is indeterminant

From the prompt, the statement under question was the following:

$$\sin(x) = \sqrt{1 - \cos^2(x)}$$

To determine if this statement is an identity, we must show that the two quantities are equal for all meaningful values of x. One way of comparing both quantities is to plot them on the coordinate graph as seen below:



The statement is true when both the left and right sides have the same sign. However, the value for $\cos^2$ (x) will always be positive and less than 1 so the graph of g(x) will also always be positive, even when sin(x) is negative, so these values of x which make sin(x) prove that the statement is not an identity.