**Situation: Exponential Rules**

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**Last Revised on August 1, 2013 by Sydney Roberts**

## Prompt

Students in an algebra class have just finished a unit on exponential powers, including standard exponential rules and negative exponents. In completing a sheet of true/false questions, most of the students have classified the statement at the right as false. $2^{17} + 2^{17} = 2^{18}$

## Commentary

The teacher must be aware of the laws of exponents and possible misconceptions surrounding them. Essentially, exponents involve multiplication, which involves addition. Both of these basic operations revolve around the field axioms. Because of this, the situation is broken into four mathematical foci that take the reader from the axiomatic properties of the real numbers, to the definition of multiplication, and finally to the notation and laws of exponents. The last focus pertains specifically to the summation presented in the prompt.

## Mathematical Foci

### Focus 1

*Multiplication is a well-defined operation between real numbers that takes 2 or more real numbers, and produces a single real number known as the product.*

Within the real numbers, there are three axiomatic properties that pertain to multiplication as a binary operation within a field: The Commutative Law, The Associative Law, and the Distributive Law.

The Commutative Law:

$a \cdot b = b \cdot a \quad \forall\ a, b \in \mathbb{R}$ (∀ symbolizes "for all").

The Associative Law:

$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall\ a, b, c \in \mathbb{R}$

The Distributive Law:

$a \cdot (b + c) = ab + ac \quad \forall\ a, b, c \in \mathbb{R}$

Existence of identity elements:

$a + 0 = a \quad \forall\ a \in \mathbb{R}$
$a \cdot 1 = a \quad \forall\ a \in \mathbb{R}$

Existence of inverse elements:

$$a + (-a) = 0 \quad \forall\, a \in \mathbb{R}$$
$$a \times \frac{1}{a} = 1 \quad \forall\, a \in \mathbb{R}$$

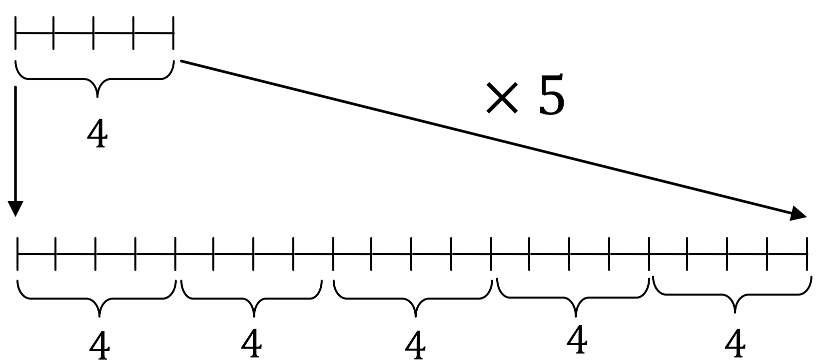
These axiomatic properties are important for understanding and deriving generalizations about multiplication across the real numbers.

**Focus 2**

*At its core, multiplication is repeated addition and can be thought of as "scaling" one element by another element.*

Essentially, understanding multiplication as repeated addition can be seen directly from the distributive law. To see this, it would be best to walk through an example. $4 \times 5$ is a simple multiplication problem involving two real numbers. Note that $5 = 1 + 1 + 1 + 1 + 1$ and hence $4 \times 5 = 4(1 + 1 + 1 + 1 + 1)$. Using the distributive law, $4(1 + 1 + 1 + 1 + 1) = 4 + 4 + 4 + 4 + 4 = 20$. Hence $4 \times 5$ is the same as adding four together five times. According to the commutative law, $4 \times 5$ is the same as $5 \times 4$ and is the repeated addition of adding five together four times.

This is where the idea of "scaling" one element by another comes from.  In the case, the number "4" is being scaled by the number "5"or vice versa.




In general, the product of 2 real numbers $a \times b$ can be evaluated as follows:

$$a \times b = \underbrace{a + a + \cdots + a + a}_{b\ times} = \underbrace{b + b + \cdots + b + b}_{a\ times}$$

It's also important to note that the distributive law states that an element can be distributed into a summation of real numbers, and the equality symbol within the law allows a real number to be "factored out" of a summation of real numbers where each summand has a common divisor. For example: $2 + 2 + 2 + 2$ is a summation of real numbers where each summand has a common divisor of two. Hence, a two can be factored out of each summand and result in $2+2+2+2=2(1+1+1+1)=2(4)=8$. More generally,

$$ax_1 + ax_2 + ax_3 + \cdots + ax_n = a(x_1 + x_2 + x_3 + \cdots + x_n)\ \forall\ a, x_{1,} \cdots, x_n \in \mathbb{R}$$

**Focus 3**

*Exponent notation is $x^n$ where x $\in \mathbb{Z} - \{0\}$ and n $\in \mathbb{Z}$. In this notation, x is generally referred to as the base, and n is the exponent. This notation represents repeated multiplication of the element x a total of n times.*

Historically, exponent notation took on many changes before becoming what we think of as "modern notation." This modern notation was brought about by Descartes in 1637, but only referred to positive integers as the exponents. Negative exponents were not used until 1676 by Newton.

For the domains of x and n as stated above, the notation $x^n$ can be defined as follows:

$$x^n = \begin{cases} 1 & \text{If } n = 0 \\ \underbrace{x \times x \times \cdots \times x}_{n\ times} & \text{If } n > 0 \\ (x^{-1})^m & \text{If } n < 0 \text{ let } n = -m \text{ where } m > 0 \text{ and} \\ & x^{-1} \text{ is the multiplicative inverse of } x \end{cases}$$

Note: $x^n$ could be defined in other domains besides those that are stated above. However, due to the scope of the prompt, this is how $x^n$ will be defined for the rest of the situation. For more information about how $x^n$ can be defined using other domains for x and n, please refer to http://jwilson.coe.uga.edu/Situations/Situation.21/SITUATION_21_expru070510.pdf .

Then two laws of exponents exist (let , $b \in \mathbb{Z}$ ):

1. $(x^a)^b = x^{a \cdot b}$
2. $x^a \cdot x^b = x^{a+b}$

The proofs of these two laws involve several cases, but they are intuitively clear when *a* and *b* are positive. Because exponents are essentially repeated multiplication, it can be seen that

$$(x^a)^b = x^a \times x^a \times \cdots \times x^a = \underbrace{\underbrace{(x \times x \times \cdots \times x)}_{a\ times} \times \underbrace{(x \times x \times \cdots \times x)}_{a\ times} \times \cdots \times \underbrace{(x \times x \times \cdots \times x)}_{a\ times}}_{b\ times}$$

$$= \underbrace{(x \times x \times \cdots \times x)}_{a \cdot b\ times} = x^{a \cdot b}$$

and that $x^a \cdot x^b = \underbrace{(x \times x \times \cdots \times x)}_{a\ times} \cdot \underbrace{(x \times x \times \cdots \times x)}_{b\ times} = \underbrace{(x \times x \times \cdots \times x)}_{a+b\ times} = x^{a+b}$.

These exponent rules allow large exponents to be broken into smaller pieces. For example,

$$2^{18} = (2^9)^2 = (2^3)^6 = 2^4 \cdot 2^{14} = 2^1 \cdot 2^{17}, \text{ etc.}$$

**Focus 4**

*If the expression $x^n$ is added together $x$ times, then the resulting sum will be $x^{n+1}$ $\forall$ x, n $\in \mathbb{R}$.*

The summation presented in the prompt was $2^{17} + 2^{17}$. This summation is "special" in the sense that it is a power with base 2, having the same exponent, being added together 2 times. Then the result was $2^{17+1} = 2^{18}$. Consider another example, $3^2 + 3^2 + 3^2$. This is a power with base 3, having the same exponent, being added together 3 times. Using the idea of multiplication as repeated addition, we can see that $3^2 + 3^2 + 3^2 = 3 \times 3^2 = 3^{2+1} = 3^3$.

In general, $\underbrace{x^n + x^n + \cdots + x^n}_{x\ times} = x \times x^n = x^{n+1}$.

**Post-Commentary**

It is important to note that there are no exponent laws dealing with the addition of exponential expressions with different exponents. However, it is easy to derive certain generalizations with exponential expressions of the same base. Consider the example of $3^2 + 3^7$. Then,

$$3^2 + 3^7 = 3^2(1 + 3^5).$$

This follows directly from the distributive law. In general, when adding exponential expressions of the same base, the expression with the smallest exponent can always be factored out of all the summands.

**References:**


http://jeff560.tripod.com/operation.html