

## Situation: Numbers Raised to the Zero Power

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### Prompt

While reviewing properties of exponents in a middle school classroom, the teacher reminds the students that any number raised to the power of zero is equal to one. A student raises their hand and states, "I don't understand why that is true."

### Commentary

The prompt is a reasonable question for a student to ask. Instead of being told a definition, it's best to have an understanding as to why that definition is true. Essentially, that is the point of the situation. The mathematical foci below involve understanding the truth behind why  $x^0 = 1$  by relating it to simple concepts such as patterns and recognizing these patterns as limits. Another focus involves using exponential rules to explain why  $x^0 = 1$ . Lastly, a focus is included that deals with the special case of zero raised to the power of zero.

### Mathematical Foci

#### Focus 1

*Using patterns is a helpful method to recognizing why  $x^0 = 1$  is true.*

Certain patterns can be seen in mathematics that sometimes help to show something is true, simply because it follows the pattern. Consider the list of numbers

$$\begin{aligned} & \dots, 4^1, 4^2, 4^3, 4^4, 4^5, \dots \\ & = \dots, 4, 16, 64, 256, 1024, \dots \end{aligned}$$

When reading the list from left to right, each number is multiplied by 4. When reading the list right to left, each number is divided by four. Therefore, it seems natural that the number preceding  $4^1$  would be one. In other words,  $4^0 = 1$  follows this pattern.

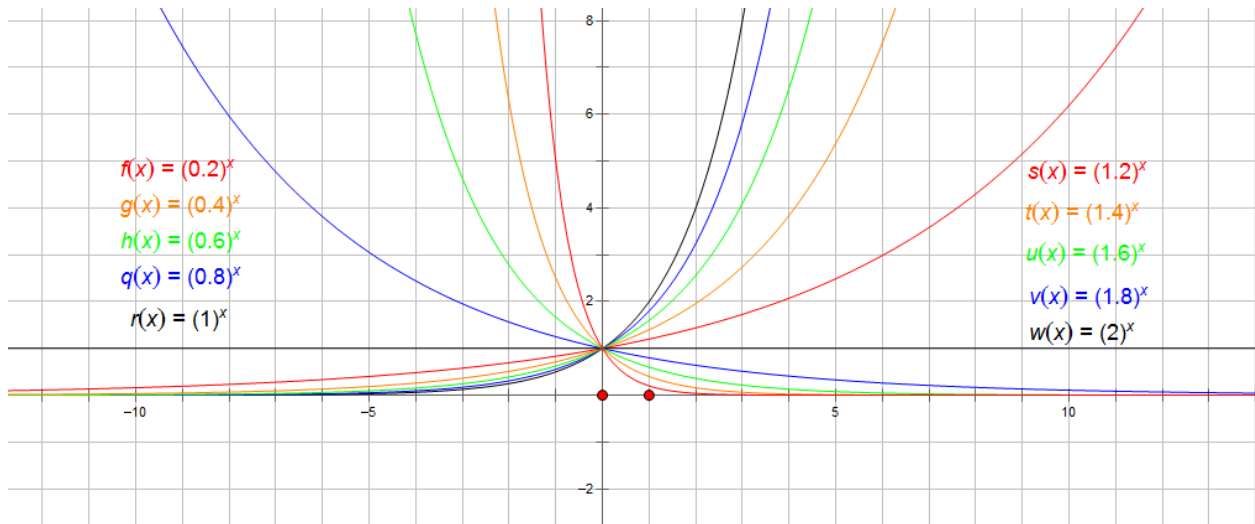
This pattern works for negative numbers also.

$$\begin{aligned} & \dots, (-4)^1, (-4)^2, (-4)^3, (-4)^4, (-4)^5, \dots \\ & = \dots, -4, 16, -64, 256, -1024, \dots \end{aligned}$$

Again, moving left to right, each number is multiplied by -4 and moving right to left, each number is divided by -4.

However, what happens with the pattern  $\dots, 0^1, 0^2, 0^3, 0^4, 0^5, \dots$ ? Of course moving left to right through the pattern holds, but moving right to left would use division by zero. This would bring about undefined numbers, but we know zero raised to the power of 1, 2, etc. are defined. So this pattern argument would not hold when  $x = 0$ .

Another way this pattern can be explained is by evaluating  $\lim_{n \rightarrow 0} x^n$  where  $x \in \mathbb{R} > 0$ . This limit converges to 1 and can be shown using a graph like the one below.



Notice that all of the functions converge at the point (0,1).

## Focus 2

Exponent Laws can be used to show why  $x^0 = 1 \quad \forall x \in \mathbb{R} - \{0\}$ .

Several laws of exponents exist and are listed below where  $x$  and  $y$  are real numbers other than zero, and  $a$  and  $b$  are real numbers. (It is important to note here that these rules do not hold when  $x$  is zero. This will be considered a special case that involves more discussion later on in Focus 3.)

1.  $x^a \cdot x^b = x^{a+b}$
2.  $\frac{x^a}{x^b} = x^{a-b}$
3.  $(x^a)^b = x^{ab}$
4.  $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
5.  $x^{-a} = \frac{1}{x^a}$
6.  $\left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^a$

In order to show that  $x^0 = 1$ , the second law will be used. Consider the example below:

$$\frac{4^6}{4^3} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4} = 4 \times 4 \times 4 = 4^3 = 4^{6-3}$$

This follows directly from the second law. However, what if  $a=b$  in the second law?

$\frac{4^6}{4^6} = 1$  simply because any number divided by itself is one. Now use the second law of exponents.

$$\frac{4^6}{4^6} = 4^{6-6} = 4^0 = 1$$

This can be generalized to numbers beyond 4. Consider  $x^0$  where  $x$  is any real number besides zero. Then  $x^0 = x^{n-n} = \frac{x^n}{x^n} = 1$  for all real numbers  $n$ . Hence,  $x^0 = 1$  for all real numbers  $x$ , not including  $x=0$ .

Another way to show this is true is by considering the example where  $x = 2$ ,  $a = -3$ , and  $b = 3$ . Then  $2^{-3} \cdot 2^3 = \frac{1}{2^3} \cdot 2^3 = \frac{1}{8} \cdot 8 = 1$ . However, this is also true by using the first rule:  $x^a \cdot x^b = x^{a+b}$  and then  $2^{-3} \cdot 2^3 = 2^{-3+3} = 2^0$ . But we also know that  $2^{-3} \cdot 2^3 = 1$ . Hence,  $2^0 = 1$ . Again, this can be generalized to show that for any real number  $x$  (besides 0),  $x^0 = x^{-n+n} = x^{-n} \cdot x^n = \frac{1}{x^n} \cdot x^n = 1$  for all real numbers  $n$ .

### Focus 3

*Zero raised to the power of zero is an indeterminate form.*

This special case of  $0^0$  has caused many debates between mathematicians. Does  $0^0 = 1$  or  $0$ ? Some say  $0^0$  should equal 1 since any number raised to 0 is one. Other says  $0^0$  should equal 0 since 0 raised to any number is zero. Because of this question, it is said that the value of  $0^0$  is indeterminate. However, sometimes  $0^0$  will be evaluated as 1 for convenience.

### References

<http://mathforum.org/dr.math/faq/faq.number.to.0power.html>

<http://mathworld.wolfram.com/Zero.html>