

Situation 1: Congruent Triangles vs. Similar Triangles

Prepared at the University of Georgia EMAT 6500

Date last revised: July 24th, 2013

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Prompt:

A teacher in a high school Analytic Geometry class is teaching a lesson on how to use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. In the middle of the lesson a student asked, “What is the difference between congruent triangles and similar triangles?”

Commentary:

The set of mathematical foci below bring attention to the background knowledge that one must have in order to understand the concepts of similarity and congruence. The first focus highlights the axiomatic approach taken by Euclid and David Hilbert, which is ultimately the foundation of plane geometry. In order to grasp the concept of congruence and similarity not only in triangles but also in other geometric figures, one must first be familiar with the Euclidean distance formula derived from the Pythagorean Theorem (second focus). The other three foci that follow concentrate on proving congruence and similarity in triangles by using the geometric properties of triangles and angles.

Mathematical Foci:

Focus 1: *This focus highlights the congruence propositions presented in Euclid’s work along with the congruence axioms of Plane Geometry developed by David Hilbert.*

Euclid’s “Elements” of geometry, written around 300 B.C. may be one of the most famous mathematical works of all time which served for thousands of years as a foundation of mathematics education. The “Elements” is made up of 13 books which cover many different mathematical concepts from plane geometry to number theory. In book 1, Euclid developed the concept of congruence within the propositions listed below. However, he did not explicitly use the term “congruent” he rather used the word “equal”.

Euclid’s Propositions of Congruence:

Proposition 4: If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. (Side – Angle – Side)

Proposition 8: If two triangles have the two sides equal to two sides respectively, and also have the base equal to the base, then they also have the angles equal which are contained by the equal straight lines. (Side – Side – Side)

Proposition 26: If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle. (Angle – Angle – Side or Side – Angle - Angle)

Euclid’s “Element’s” introduced the axiomatic method of geometry; however, there were many shortcomings within Euclid’s procedures. His work and ideas seemed to be somewhat incomplete, which lead David Hilbert to publish a new set of axioms and propositions in his book, “Grundlagen der Geometrie,” which translates to “Foundations of Geometry” and was published in 1898. Hilbert developed the following axioms of congruence:

Hilbert's Axioms of Congruence:

Axiom 1: If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a unique point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.

Axiom 2: If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

Axiom 3: If B lies between A and C, and B' lies between A' and C', $AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

Axiom 4: Given any $\sphericalangle BAC$ and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a unique ray $\overrightarrow{A'C'}$ on a given side of line $\overrightarrow{A'B'}$ such that $\sphericalangle BAC \cong \sphericalangle B'A'C'$.

Axiom 5: If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle B \cong \sphericalangle C$, then $\sphericalangle A \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

Axiom 6: (SAS) If two sides and the included angle of one triangle are congruent respectively to two sides and the included angle of another triangle, then the two triangles are congruent.

The systematic and axiomatic approach to geometry is essentially a list of postulates and propositions that describe the properties of points, lines, line segments, etc. The axiomatic approach differs from analytic geometry in such a way that the axiomatic approach does not deal with the Cartesian plane. This will be discussed further in the next focus.

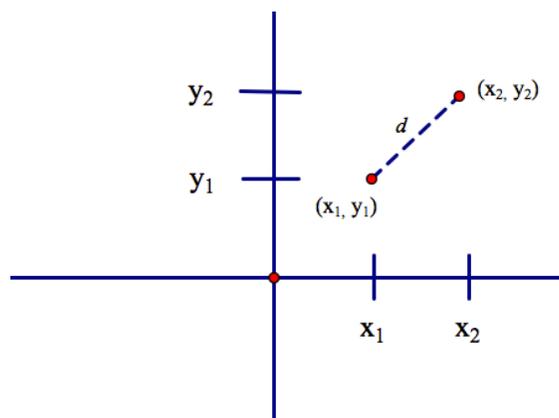
Focus 2: *The Euclidean Distance d between the points (x_1, y_1) and (x_2, y_2) is given by*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

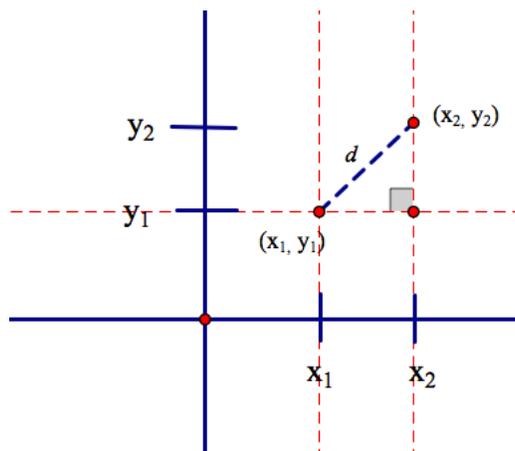
In analytic geometry, congruence is defined as follows: Two mappings of figures onto one Cartesian coordinate system are congruent if and only if, for any two points in the first mapping, the Euclidean distance between them is equal to the Euclidean distance between the corresponding points in the second mapping.

Deriving the Distance Formula:

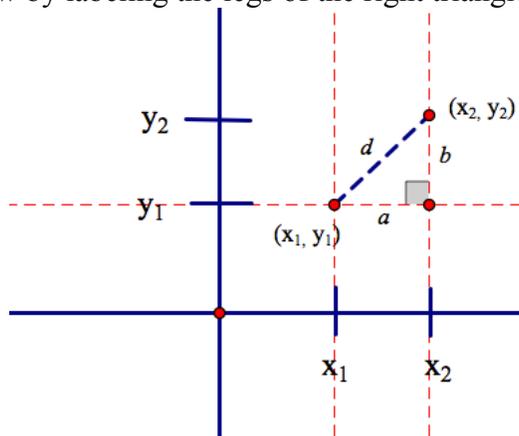
Start by drawing a line segment anywhere on a coordinate grid. Label the endpoints of the segment as (x_1, y_1) and (x_2, y_2) .



Construct a right triangle by dropping a perpendicular line through the point (x_2, y_2) and constructing another perpendicular line through the point (x_1, y_1) as shown below.



Then, follow by labeling the legs of the right triangles as a and b .



Now, using the triangle you have constructed, apply the Pythagorean Theorem to find the distance d between the points (x_1, y_1) and (x_2, y_2) . Thus, we have the following equation:

$$d^2 = a^2 + b^2$$

However, we know that the length of side a is equal to the absolute value of the horizontal distance between x_1 and x_2 (i.e. $a = |x_2 - x_1|$) and the length of side b is equal to the absolute value of the vertical distance between y_1 and y_2 (i.e. $b = |y_2 - y_1|$). Thus, substituting those values into the Pythagorean Theorem we get:

$$d^2 = (|x_2 - x_1|)^2 + (|y_2 - y_1|)^2$$

$$\sqrt{d^2} = \sqrt{(|x_2 - x_1|)^2 + (|y_2 - y_1|)^2}$$

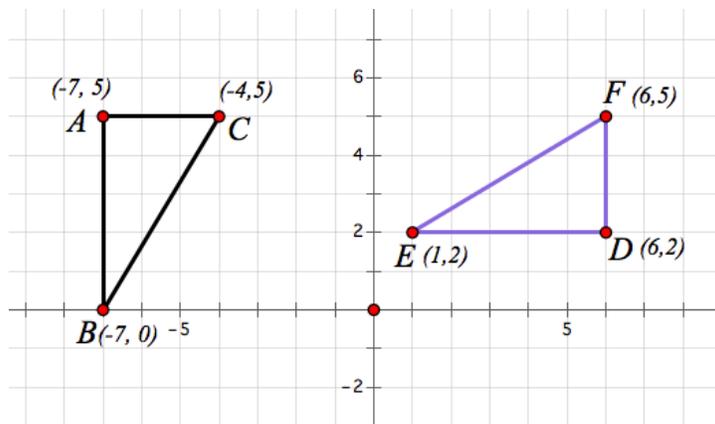
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

QED

Focus 3: *The Concept of Congruence: Congruent triangles have the same shape and size. The concept of congruence applies to figures of any type; however, in this focus, we will concentrate on congruent triangles.*

Congruent triangles have the same size and the same shape. Congruent triangles are similar figures with a ratio of similarity of 1, that is $\frac{1}{1}$. The corresponding sides and corresponding angles of congruent triangles are equal. In the Common Core Georgia Performance Standards, it is heavily advocated to use coordinate geometry for proofs. Thus, the following is an example of proving triangles are congruent in a coordinate plane.

Example 1: Prove the following triangles are congruent.



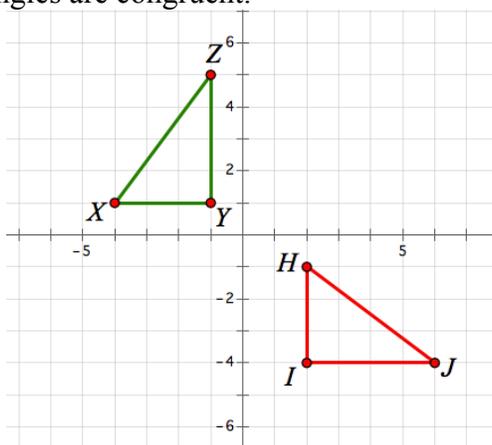
Since $AC = 3$ and $DF = 3$, $\overline{AC} \cong \overline{DF}$. Since $AB = 5$ and $DE = 5$, $\overline{AB} \cong \overline{DE}$. We can use the Distance Formula to find the lengths of BC and EF .

$$\begin{aligned}\overline{BC} &= \sqrt{(-4 - (-7))^2 + (5 - 0)^2} \\ \overline{BC} &= \sqrt{3^2 + 5^2} = \sqrt{34}\end{aligned}$$

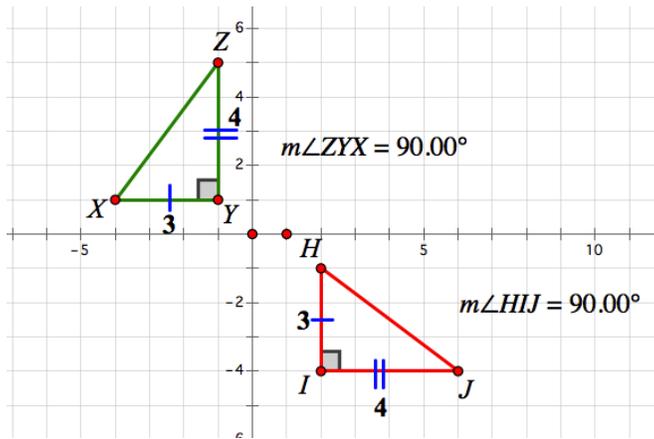
$$\begin{aligned}\overline{EF} &= \sqrt{(6 - 1)^2 + (5 - 2)^2} \\ \overline{EF} &= \sqrt{5^2 + 3^2} = \sqrt{34}\end{aligned}$$

Because $\overline{BC} = \sqrt{34}$ and $\overline{EF} = \sqrt{34}$, $\overline{BC} \cong \overline{EF}$. Therefore, all three pairs of corresponding sides are congruent, so $\triangle ABC \cong \triangle DEF$ by the SSS Congruence Postulate.

Example 2: Prove the following triangles are congruent.

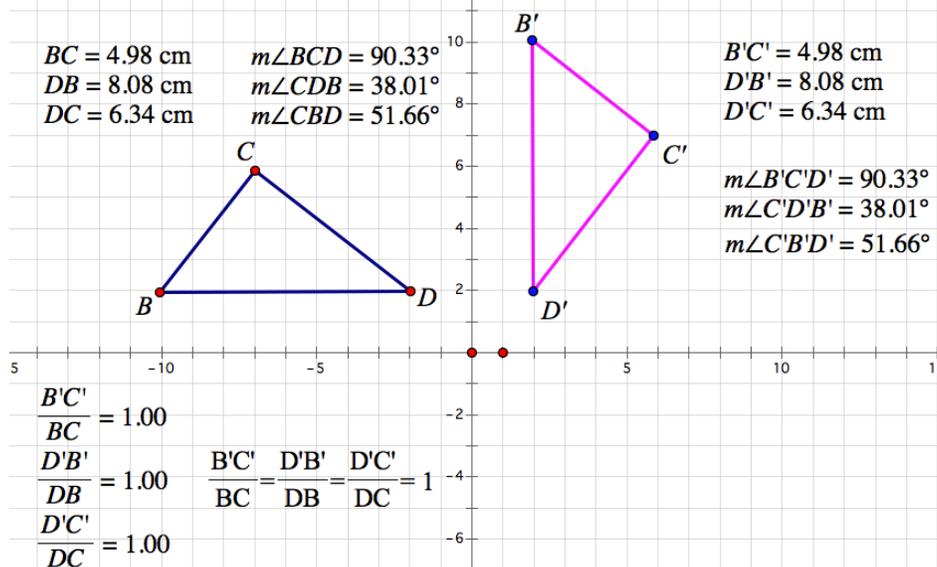


Since $XY = 3$ and $HI = 3$, $\overline{XY} \cong \overline{HI}$. Since $YZ = 4$ and $IJ = 4$, $\overline{YZ} \cong \overline{IJ}$. We can use the Distance Formula to find the lengths of XY and HJ or just recognize that it is a perfect right triangle. Thus, $m \sphericalangle XYZ = m \sphericalangle HIJ$.



Therefore, $\Delta XYZ \cong \Delta HIJ$ by the SAS Congruence Postulate

An important concept to remember is that the factor for congruent triangles is always equal to 1. For example, take a look at the Geometer's Sketchpad example below. The corresponding sides are congruent and the corresponding angles are congruent. In addition, the proportion of the corresponding sides is equal to 1.



If you were to change the size and shape of a pair of triangles by dragging vertex B, C, or D, the corresponding angles ($\sphericalangle B'$, $\sphericalangle C'$ and $\sphericalangle D'$) would remain congruent. Since the sum of three angles must be 180° according to the Angle Sum Theorem, the third pair of corresponding angles must also be congruent when the first two pairs of corresponding angles are congruent.

Triangle Congruence Theorems:

1. Side – Side – Side (SSS): If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.

Click here (http://www.youtube.com/watch?v=PqkVf0TR_1Y) for Euclid's proof of this Congruence Theorem

2. Side – Angle – Side (SAS): If two sides and the included angle of a triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.

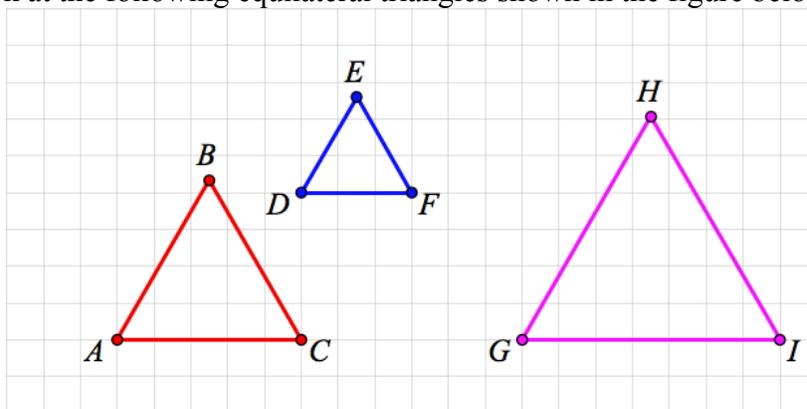
Click here (http://www.youtube.com/watch?v=sk2dL_kitcE) for Euclid's proof of this Congruence Theorem

3. Angle – Side – Angle or Side – Angle – Angle (ASA or SAA): if two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent. (The included side is the side between the vertices of the two angles.)
 Click here (http://www.youtube.com/watch?v=S-io0P5_YWA) for Euclid's proof of this Congruence Theorem

4. Hypotenuse – Leg (HL): If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, the two triangles are congruent. It is important to remember that this postulate only applies to right triangles!
 Click here (<http://www.youtube.com/watch?v=-x-raNIGjMo>) to learn more about the Hypotenuse – Leg Congruence Theorem

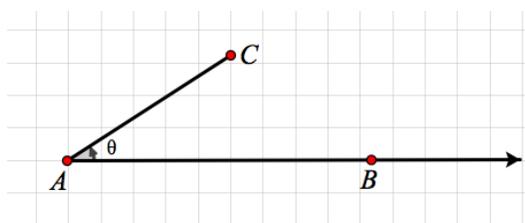
Triangle Non – Congruencies:

1. Angle – Angle – Angle (AAA) is not a congruency theorem.
 Proof: Look at the following equilateral triangles shown in the figure below:

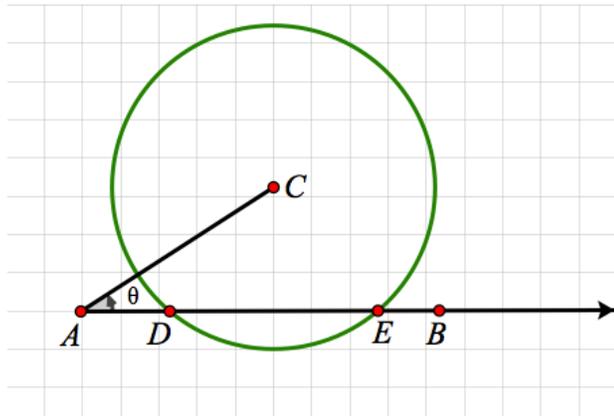


All three triangles are equilateral triangles, so they all have the same angles (each angle is 60 degrees), but they are not congruent because they have different side lengths. The blue triangle has a side length of 3, the red triangle has a side length of 5, and the pink triangle has a side length of 7.

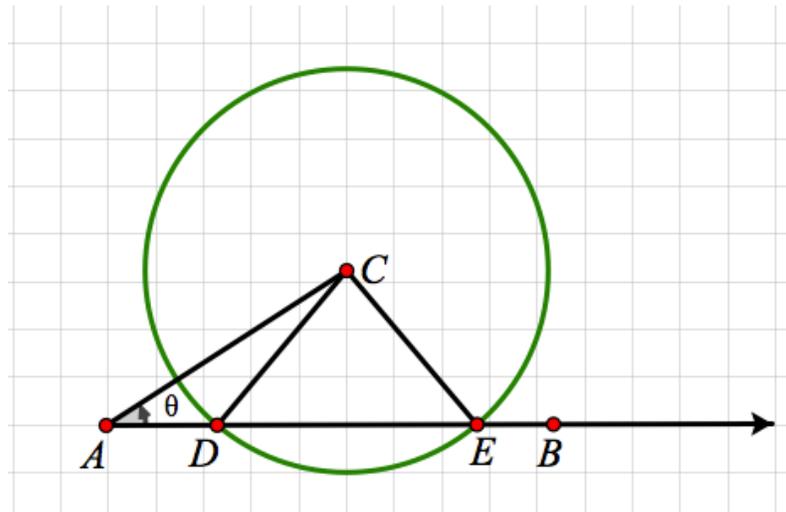
2. Side – Side – Angle (SSA) is not a congruency theorem.
 Proof: Let's begin by drawing a ray \overrightarrow{AB} , and a segment \overline{AC} with a fixed angle θ .



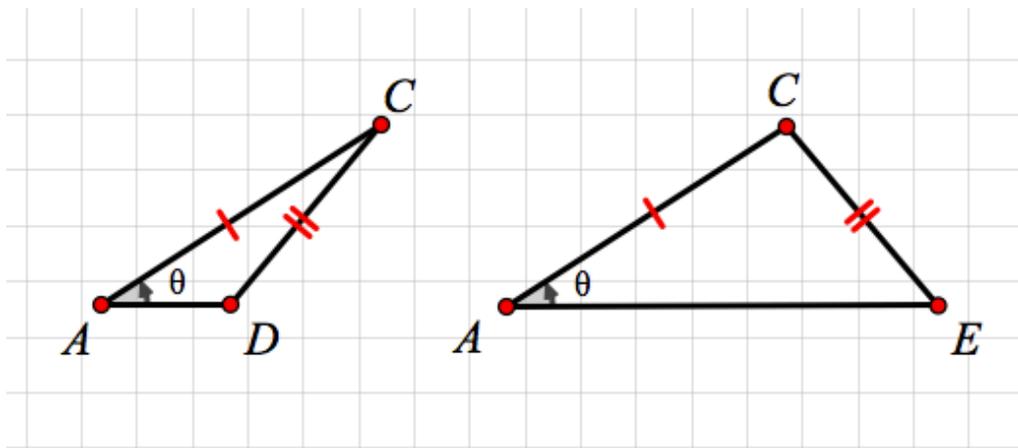
Then construct a circle centered at C using a radius such that circle intersects ray AB in two places. Label the points of intersections D and E.



Draw segments \overline{AC} , \overline{DC} , and \overline{CE} .



Notice that using the radius of the circle, $\overline{DC} \cong \overline{CE}$. Now, let's redraw the triangles as shown below:

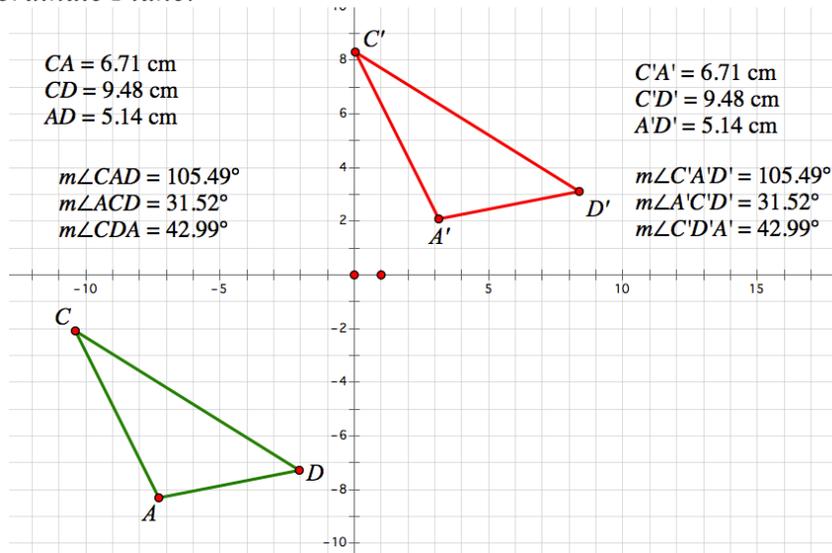


Notice that $\triangle ADC$ is an obtuse triangle. We know angle A is congruent to angle A because it was a fixed angle from the beginning, and we know AC is congruent to AC because it was also a fixed side. Now, since CE is a radius of the circle, it must be congruent to CD . However, we have created two different triangles. Thus, SSA is ambiguous, because it does not give you enough information to determine that the triangles are exactly the same. This is essentially a counterexample, thus, showing that SSA is not a congruency theorem.

Focus 4: Defining Congruence in terms of rigid motions.

In the field of Euclidean geometry, rigid motion is any combination of reflection, translation, and rotation preserving angle measure and side length. These transformations are known as congruence transformations because they alter a set of points in some way, but the points' relationship with one another stays the same. Thus, there exists an isometry between the pre – image and the image. Thus, the image created after a transformation takes place is congruent to the pre – image. However, dilation is not an isometry since it either shrinks or enlarges a figure, so it does not preserve angle measure or side length.

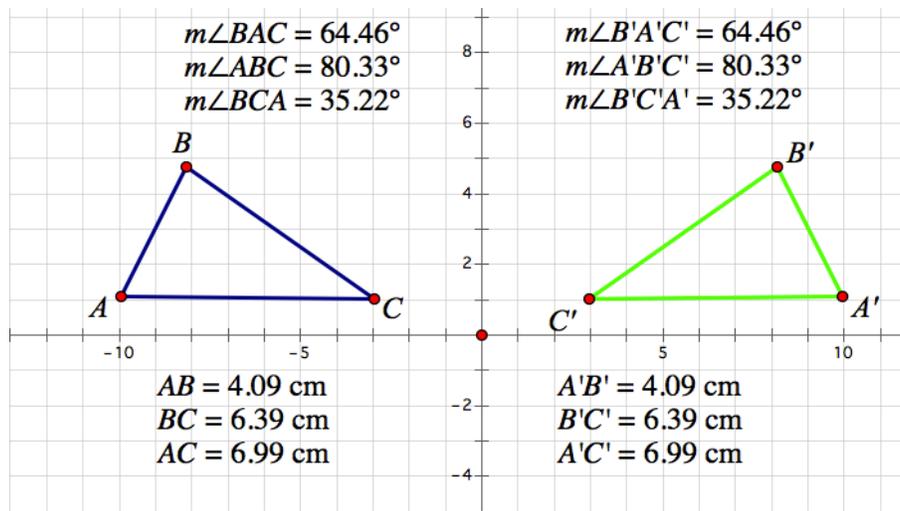
Translations in the Coordinate Plane:



Properties of Translations:

- Translated figures are congruent to the original figure (In the example shown above, $\triangle ACD \cong \triangle A'C'D'$)
- Angles translate to congruent angles:
 $(\sphericalangle CAD \cong \sphericalangle C'A'D', \sphericalangle ACD \cong \sphericalangle A'C'D', \text{ and } \sphericalangle CDA \cong \sphericalangle C'D'A')$
- Line segments (side lengths) translate to congruent line segments:
 $(\overline{CA} = \overline{C'A'}, \overline{CD} = \overline{C'D'}, \text{ and } \overline{AD} = \overline{A'D'})$

Reflections in the Coordinate Plane:

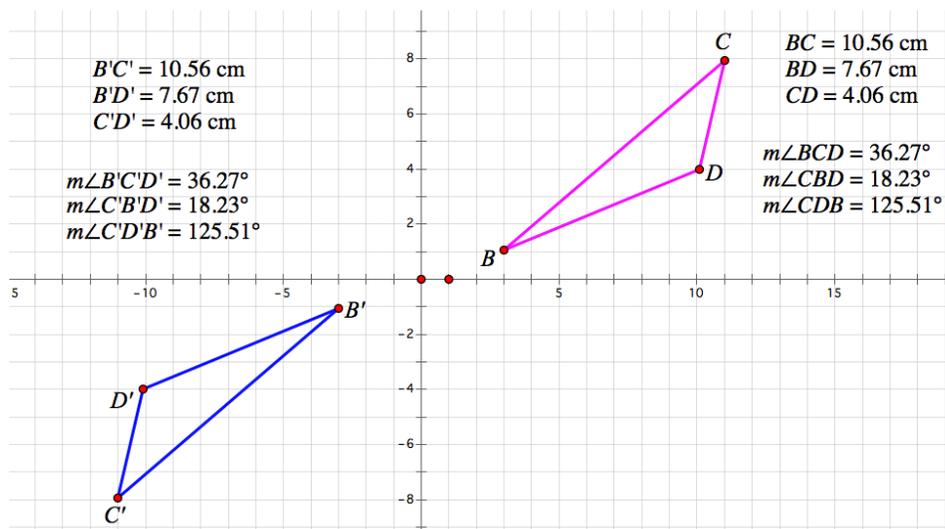


The concept of distance can be applied with reflections since the line of reflection (in this example the y – axis is the line of reflection) is equidistant from both A and A' , B and B' , and C and C' . Thus, the line of reflection is directly in the middle of every point on each triangle. Thus, the pre – image is always the same distance away from the line of reflection as the image.

Properties of Reflections:

- Reflected figures are congruent to the original figure (In the example shown above, $\Delta ABC \cong \Delta A'B'C'$)
- Angles translate to congruent angles:
 $(\sphericalangle BAC \cong \sphericalangle B'A'C', \sphericalangle ABC \cong \sphericalangle A'B'C', \text{ and } \sphericalangle BCA \cong \sphericalangle B'C'A')$
- Line segments (side lengths) translate to congruent line segments:
 $(\overline{AB} = \overline{A'B'}, \overline{BC} = \overline{B'C'}, \text{ and } \overline{AC} = \overline{A'C'})$

Rotations in the Coordinate Plane:



Triangle BCD has been rotated 180 degrees counterclockwise about the origin.

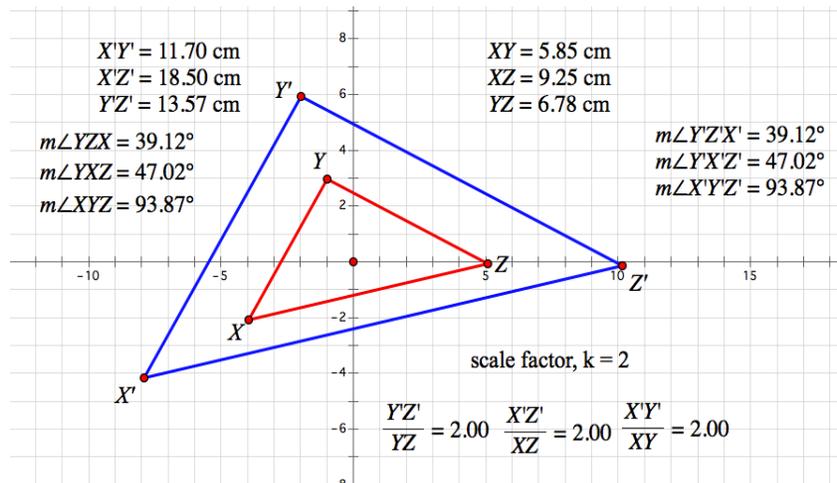
Properties of Rotations:

- Rotated figures are congruent to the original figure (In the example shown above, $\Delta BCD \cong \Delta B'C'D'$)
- Angles translate to congruent angles:
 $(\sphericalangle BCD \cong \sphericalangle B'C'D', \sphericalangle CBD \cong \sphericalangle C'B'D', \text{ and } \sphericalangle CDB \cong \sphericalangle C'D'B')$
- Line segments (side lengths) translate to congruent line segments:
 $(\overline{BC} = \overline{B'C'}, \overline{BD} = \overline{B'D'}, \text{ and } \overline{CD} = \overline{C'D'})$

Focus 5: *The Concept of Similarity: Triangles are similar if they have the same shape, but not necessarily the same size.*

Non –Rigid Transformation: Dilations in the Coordinate Plane

A dilation enlarges or reduces the size of a figure; thus, the pre – image and image of a dilation are not congruent, but they are similar. A dilation of scalar factor k whose center of dilation is the origin may be written as: $D_k(x, y) = (kx, ky)$. If the scale factor, k , is greater than 1, the image is an enlargement or a stretch. If the scale factor, k , is between 0 and 1, the image is a reduction or a shrink.



In the above figure, triangle XYZ has been dilated by a scale factor of 2 to create triangle $X'Y'Z'$.

Properties of dilation:

- 1) Angle measures remain the same and are congruent under a dilation.
(i.e. $\sphericalangle YZX \cong \sphericalangle Y'Z'X'$, $\sphericalangle YXZ \cong \sphericalangle Y'X'Z'$, and $\sphericalangle XYZ \cong \sphericalangle X'Y'Z'$)
- 2) Distance is not preserved. Thus, a dilation is not an isometry because the lengths of the sides of the triangle (or the lengths of the segments) are not equal.

The above triangles are similar triangles. They are two triangles with three identical angles but not necessarily identical lengths. However, if they are not identical lengths they must have an identical scale factor for each side.

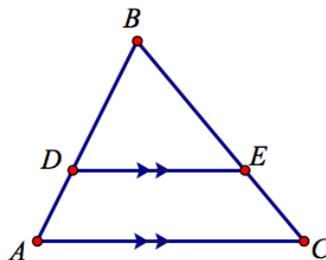
Properties of Similar Triangles:

- 1) Corresponding angles are congruent
- 2) Corresponding sides are all in the same proportion

How to determine if triangles are similar:

- 1) AA (Angle – Angle): If two angles of one triangle are congruent to two angles of another triangle, the triangles are similar.

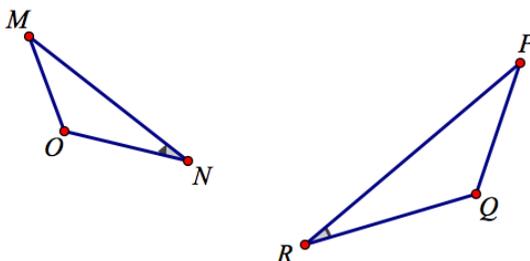
Example: Given $\overline{AB} \parallel \overline{DE}$. Prove $\triangle ABC \sim \triangle DBE$



Statement	Reason
$\overline{AB} \parallel \overline{DE}$	Given
$\sphericalangle A \cong \sphericalangle BDE$	Corresponding Angles
$\sphericalangle C \cong \sphericalangle BED$	Corresponding Angles
$\triangle ABC \sim \triangle DBE$	AA

- 2) SAS (side – angle – side): If an angle of one triangle is congruent to the corresponding angle of another triangle and the lengths of the sides including these angles are in proportion, the triangles are similar.

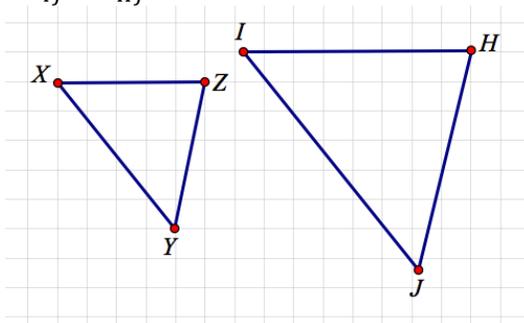
Example: Given: $\frac{MN}{PR} = \frac{ON}{QR}$, $\sphericalangle N \cong \sphericalangle R$. Prove $\Delta MNO \sim \Delta PQR$



Statement	Reason
$\frac{MN}{PR} = \frac{ON}{QR}$	Given
$\sphericalangle N \cong \sphericalangle R$	Given
$\Delta MNO \sim \Delta PQR$	SAS

- 3) SSS (side – side – side): If the three sets of corresponding sides of two triangles are in proportion, the triangles are similar.

Example: Given $\frac{XZ}{IH} = \frac{XY}{IJ} = \frac{ZY}{HJ}$. Prove $\Delta XYZ \sim \Delta IJH$



Statement	Reason
$\frac{XZ}{IH} = \frac{XY}{IJ} = \frac{ZY}{HJ}$	Given
$\Delta XYZ \sim \Delta IJH$	SSS

It is imperative to know that the SSS theorem and the SAS theorem for similarity are not the same as the SSS theorem and the SAS theorem for congruence. Similarity specifically deals with the proportion of sides.

Post Commentary:

In making mathematical statements, it is important to use correct mathematical vocabulary and terminology. Therefore, knowing the difference between similar and congruent triangles is an essential foundation for building higher mathematics, and being able to reason through proofs. The foci presented throughout this document are meant to provide teachers with some background knowledge that they should be familiar with in order to address the difference between similar and congruent triangles.

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