Situation 2: Undefined Slope vs. Zero Slope

Prepared at the University of Georgia
EMAT 6500 class
Date last revised: July 1^{st}, 2013
Nicolina Scarpelli

Prompt:

A teacher in a 9^{th} grade Coordinate Algebra class is teaching a lesson reviewing the concepts of the slope of a line before beginning a new unit that uses coordinates to prove simple geometric theorems algebraically. While reviewing the different types of slope, a student raises her hand and asks, "What is the difference between zero slope and undefined slope?"

Commentary:

The set of mathematical foci below bring attention to the background knowledge that one must have in order to understand the mathematical concept of slope. Slope is a very broad topic in mathematics and can be represented in many different ways depending on the content area. With significance placed on rate of change, its definition and misunderstandings need to be understood by teachers so that students might gain conceptual understanding of the topic. Misunderstandings include concerns with formulas and understanding slope in contextual, real – life examples. The first focus highlights the history of when and where the concept of slope was developed; the next three foci emphasize the concept of slope within algebra, geometry, trigonometry, and calculus. The last focus underscores the importance of knowing the properties of dividing by zero to address the concept of undefined slope.

Mathematical Foci:

Focus 1: *It is essential for teachers of mathematics to know the historical development of the concepts they teach because it leads to a greater insight of the "why's" and "how's" of mathematics. This first focus highlights the history of when, where, and how the concept of slope originated.*

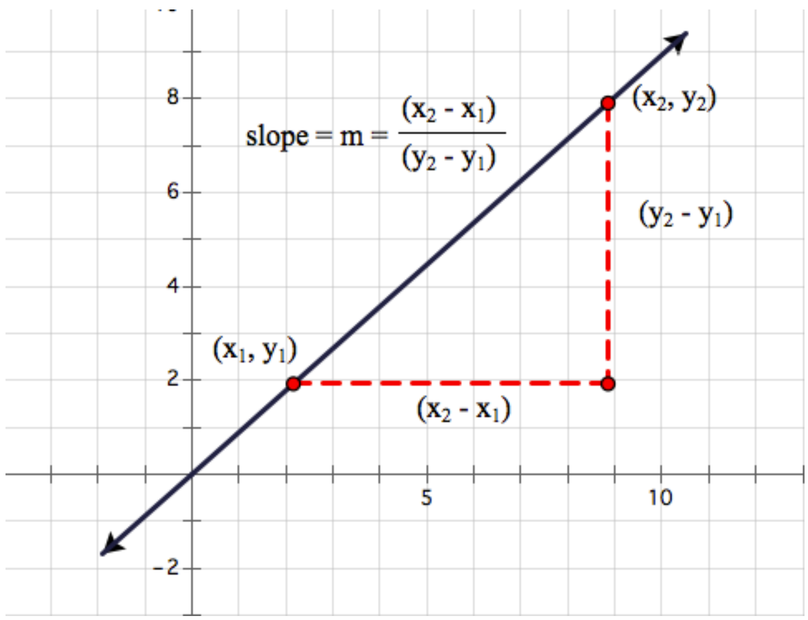
Rene` Descartes (March 31, 1596 – February 11, 1650), was a prominent and noteworthy French mathematician and philosopher. Descartes made numerous advances within the mathematics world, however, one of the most vital influences that he made in mathematics was inventing the Cartesian coordinate system that is used prominently in plane geometry and many algebra based curriculums taught in school today. According to the New World Encyclopedia, Descartes founded analytic geometry in his work titled La Gèomètrie, which was crucial to the development of calculus and analysis later solidified by Isaac Newton and Gottfried Wilhelm von Leibniz. Many writers dispute that Descartes was the first to develop the coordinate system and analytic geometry. According to Ann Gantert, author of the Amsco Geometry textbook, Pierre de Fermat (1601 – 1665) actually independently developed analytic geometry a few years before Descartes, however, Descartes was the first to publish his work. As stated by James Newman, "Fermat may have preceded Descartes in stating problems of maxima and minima; but Descartes went far past Fermat in the use of symbols, in "arithmetizing" analytic geometry by extending it to equations of higher degree. Further, Descartes fixed a point's position in the plane by assigning two numbers (now known as coordinates) by calculating the point's distance from two perpendicular lines (now known as the x and y – axis)." Therefore, analytic geometry and the discovery of the Cartesian plane are ultimately credited to Descartes' discoveries in mathematics. Since linear equations and the concept of slope are established within the eighth grade curriculum solely using proportional relationships within the coordinate plane, many mathematicians recognize Descartes as the person who invented the mathematical concept of slope along with the slope formula.

<u>Focus 2:</u> *Algebraic and Geometric Conception of Slope: The slope of a line is a measure defined by the ratio of vertical change to horizontal change or "rise over run".*

Traditional Algebra textbooks introduce the slope (or gradient) of a line as a measure defined by the ratio of vertical change to horizontal change, or more likely known as "rise over run". It is a rate of change in *y* with respect to *x* that measures the steepness of a line. In more mathematical terms, given a Cartesian plane, slope can be defined as change in the y – coordinates divided by the change in the x – coordinates. Thus, the slope of a line is defined based on the coordinates of two points.

Given two points (x_1, y_1) and (x_2, y_2):

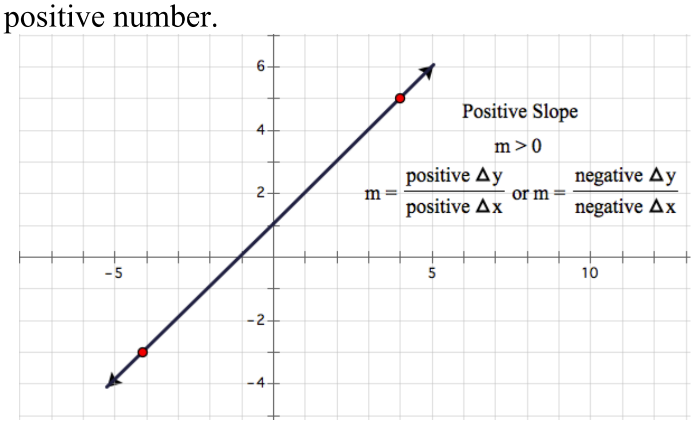
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



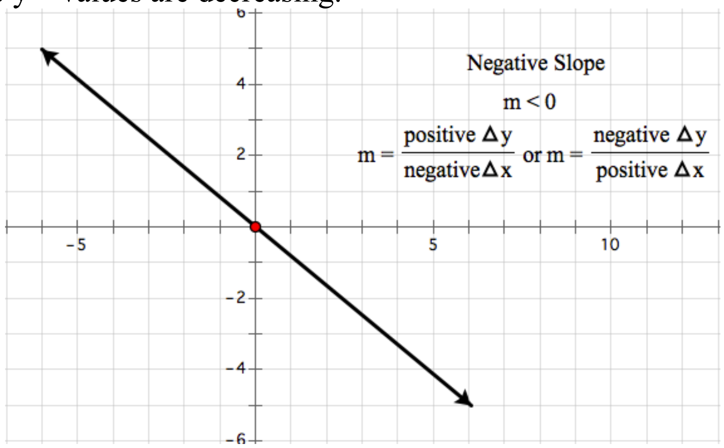
Although it doesn't matter which point you start with, being consistent is essential. For example, to calculate the slope you may use the formula $m = \frac{y_1 - y_2}{x_1 - x_2}$; however, you may not use the formula $m = \frac{y_1 - y_2}{x_2 - x_1}$. Whatever point you choose as the starting point in the numerator must be the same point you pick in the denominator (i.e. if you start with y_1 first in the numerator, you must start with x_1 in the denominator).

Slope can be positive, negative, zero, or undefined.

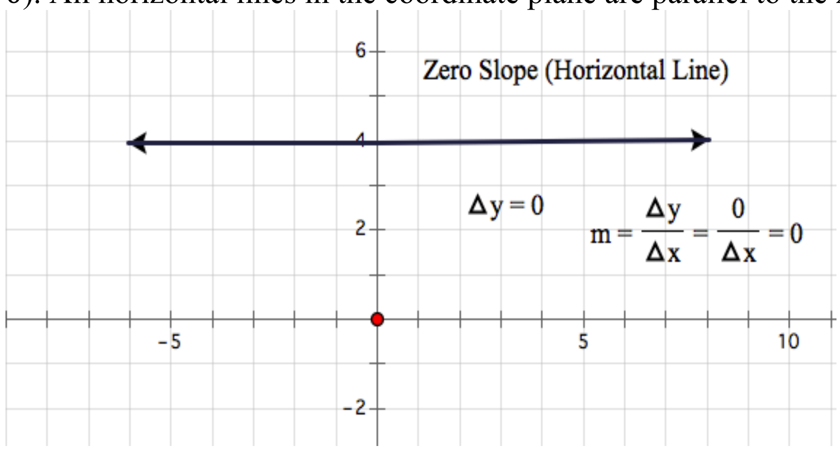
1) Positive slope means that the line is increasing from left to right or rising. Thus, as the x – values are increasing, the y – values are also increasing. A large positive number means the graph is steeper than for a smaller positive number.



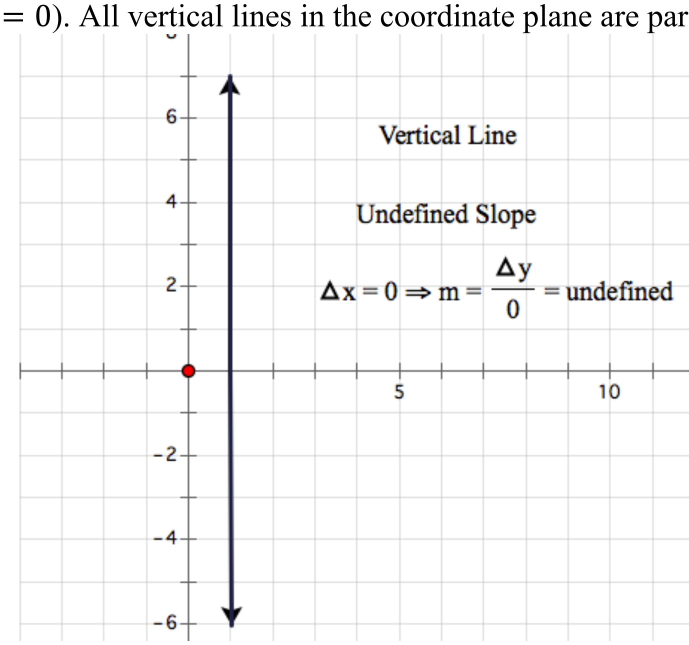
2) Negative slope means that the line is decreasing from right to left or falling. Thus, as the x – values are increasing, the y – values are decreasing.



3) Zero slope means that the line is horizontal, thus there is no change in the y – values. The y – values stay constant on a horizontal line (i.e. rise = 0 because there is no vertical change, the quantity $(y_2 - y_1) = 0$). All horizontal lines in the coordinate plane are parallel to the x – axis.



4) Undefined slope means that the line is vertical, thus there is no change in the x – values. The x – values stay constant on a vertical line (i.e. run = 0 because there is no horizontal change, the quantity $(x_2 - x_1) = 0$). All vertical lines in the coordinate plane are parallel to the y – axis.



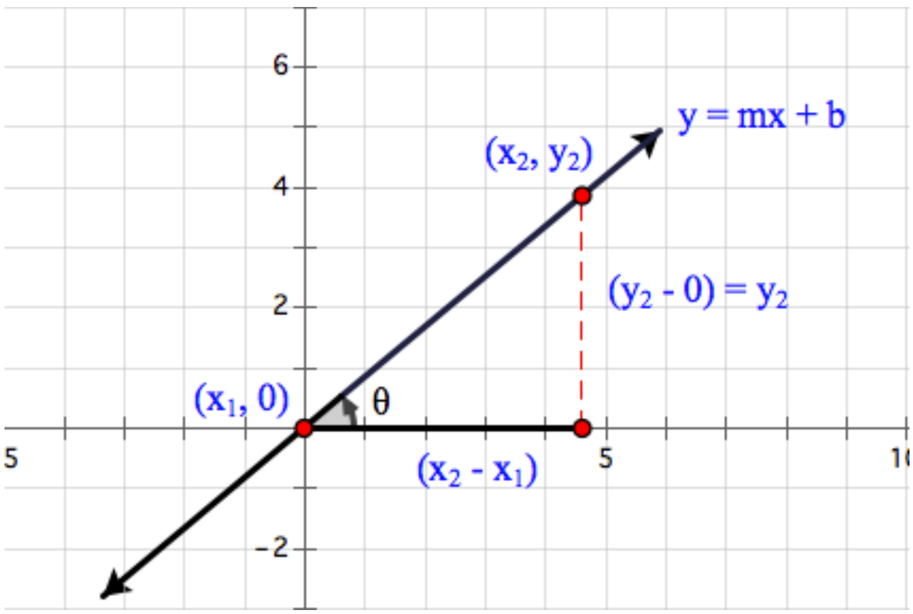
Focus 3: *Trigonometric Conception of Slope: Slope is the tangent of the angle made by a straight line with the x-axis.*

In trigonometry, the inclination of a non - horizontal line is the positive angle θ (less than π (i.e. the angle is always between 0° and 180°)) measured counterclockwise from the *x*-axis to the line. The slope of a line y = mx + b is equal to the tangent of the angle of inclination (i.e. the angle it forms with the x – axis). The following proof is found in the 8th edition Pre – calculus textbook written by Ron Larson.

Theorem: If a non – vertical line has inclination θ and slope m, then $m = \tan\theta$.

Proof: If m = 0, the line is horizontal and $\theta = 0$. So, the result is true for horizontal lines because m = 0 and $\tan(0) = 0$. If m = undefined, the line is vertical and $\theta = 90°$. So, the result is true for vertical lines because m = undefined and $\tan(\frac{\pi}{2}) = \frac{1}{0}$ = undefined.

If the line has a positive slope, it will intersect the x – axis at some point. Label this point $(x_1, 0)$, as shown in the figure below.



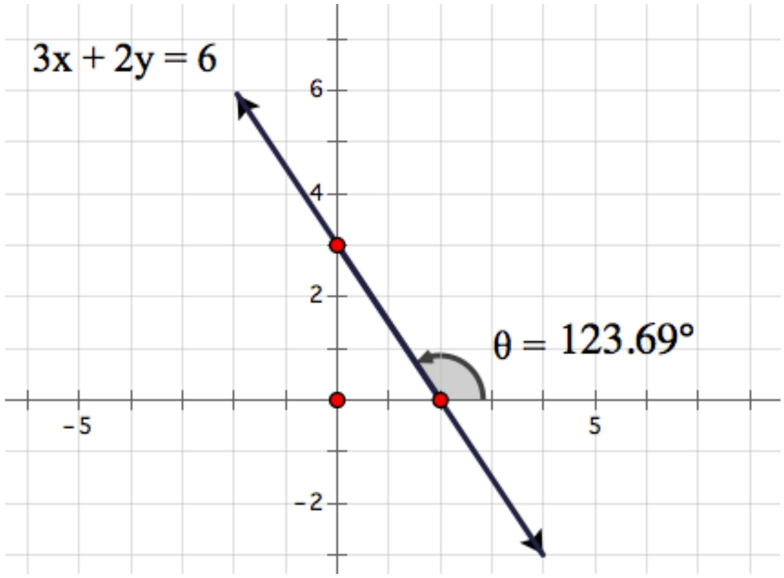
If (x_2, y_2) is a second point on the line $y = mx + b$, then the slope of the line is

$$m = \frac{y_2 - 0}{x_2 - x_1} = \frac{y_2}{x_2 - x_1} = \tan\theta$$

The case in which the line has a negative slope can be proved in a similar manner.

Example 1: Find the inclination of the line $3x + 2y = 6$

Solution: The slope of this line is $-\frac{3}{2}$, so it's inclination is determined by the equation $\tan(-\frac{3}{2})$. From the figure below, it follows that $\frac{\pi}{2} < \theta < \pi$. This means that $\theta = \pi + \tan^{-1}(-\frac{3}{2})$.



$$\theta = \pi + \tan^{-1}(-\frac{3}{2})$$

$$\theta \approx \pi + (-0.98279)$$

$$\theta \approx 2.158799$$

Thus, the angle of inclination is about 2.158799 radians or approximately 123.69 degrees.

Focus 4: *Calculus Conception of Slope: Calculus typically begins with the study of derivatives and rates of change, using slopes of lines to develop these concepts.*

(Real life examples of slope in calculus)

Focus 5: Slope in Statistics:

Focus 6: *Division by Zero*

Post Commentary:

From my experience with student teaching, I realized that my students were proficient in calculating the value of the slope (i.e. the number itself), but rather inexperienced in connecting meaning to the number. Thus, the conceptual understanding of what the slope actually means in a contextual setting is important for students to develop.

References:

http://www.merga.net.au/documents/MERJ_11_2_Stump.pdf

http://www.clarku.edu/~djoyce/trig/tangents.html

http://www.newworldencyclopedia.org/entry/Rene_Descartes#Mathematical_legacy http://www.jmap.org/JMAP/SupportFiles/Amsco_Textbooks/Geometry/Amsco_Geometry_Textbook_Chapter_6.pdf

http://4dlab.info/article_short_history_of_the_cartesian_coordinates.htm

http://books.google.com/books?id=Xn9rXyPSrzAC&pg=PA97&lpg=PA97&dq=rene+descartes+and+the+slope+formula&source=bl&ots=g7QcfoOP8k&sig=wIMbdsnbqtMjlJb4KBTXCXh2wLQ&hl=en&sa=X&ei=G9XSUdnMDYKa9gSv24HQCA&ved=0CFIQ6AEwBzgK#v=onepage&q=rene%20descartes%20and%20the%20slope%20formula&f=false