Situation: Dividing Linear Expressions

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Prompt: An Algebra II class has been examining the product of two linear expressions:

$$(ax + b)(cx + d)$$

Well into the class a student asks "What would happen if we DIVIDED one linear expression by another?"

Commentary:

The set of foci for this situation attempts to explore the basics for dividing linear expressions. The first four foci highlight the properties of long division, graphs of rational functions, vertical and horizontal asymptotes, and translations as they relate to the parent function. Rational functions will further be explored in focus five where division of higher degree polynomials is examined.

Mathematical Foci:

Mathematical Focus 1: Long (Polynomial) Division

Any two expressions can be divided by one another using a process of long division. Long (polynomial) division only works for expressions where the degree in the numerator is greater than or equal to the degree in the denominator.

When you take two linear expressions and divide one by the other you are essentially creating a rational function where you have one linear expression as the numerator (the dividend) and one linear expression as the denominator (divisor). If the divisor is not equal to zero, this process will result in a quotient and a remainder. For example, take the rational function below:

$$\frac{6x + 3}{2x + 1}$$

In this example, the linear expression *6x + 3* will be the dividend and the linear expression *2x + 1* will be the divisor. Thus, you can set up the long division expression as follows:

$$2x + 1\,\overline{)6x + 3}$$

Now, we need to ask ourselves what we need to multiply *2x + 1* to get the first term in the polynomial. In this case, that is 3. So multiply *2x + 1* by 3 and subtract the results from the first polynomial *6x + 3*. Now we have,

$$2x + 1\overline{)6x + 3}$$

$$\frac{ax+b}{cx+d}$$

In this example, the linear expression $ax + b$ will be the dividend and the linear expression $cx + d$ will be the divisor. Thus, you can set up the long division expression as follows:

$$cx+d\,\overline{)ax+b}$$

Mathematical Focus 2: Asymptotes

Vertical asymptotes are vertical lines that correspond to the zeroes of the denominator of a rational function. Horizontal asymptotes are horizontal lines that correspond to the limit of the rational function as the x value approaches positive or negative infinity.

The quotient of two linear functions results in a rational equation which can be expressed as the ratio of two polynomial functions. Since we are dealing with one variable, we can write this in the form $k(x) = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$. Even more specifically, since we are dealing with linear equations, we can express this as $y = \frac{m_1x + b_1}{m_2x + b_2}$. Let's observe the general case of the quotient of two linear functions to show the asymptotes of the equation. Let $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Solve when $h(x) = \frac{f(x)}{g(x)}$. $h(x) = \frac{m_1x + b_1}{m_2x + b_2}$. Let's set the denominator equal to zero since we know the equation is undefined at this value.

$$m_2x + b_2 = 0$$

$$m_2x = -b_2$$

$x = \frac{(-b_2)}{(m_2)}$; this is our vertical asymptote. Notice it is found by dividing the negative y-intercept of our linear function in the denominator by the slope of the linear equation in the denominator.

To solve for the horizontal asymptote allow h(x) to be represented as y to solve for the inverse.

$$y = \frac{m_1x + b_1}{m_2x + b_2}$$

$$y(m_2x + b_2) = (m_1x + b_1)$$

$$ym_2x + yb_2 = (m_1x + b_1)$$

$$ym_2x - m_1x = b_1 - yb_2$$

$$x(ym_2 - m_1) = (b_1 - yb_2)$$

$$x = \frac{b_1 - yb_2}{ym_2 - m_1}$$

Now that we've found the inverse, let's set it's denominator equal to zero to find our undefined value for x.

$$ym_2 - m_1 = 0$$

$$ym_2 = m_1$$

$y = \frac{m_1}{m_2}$; this is our horizontal asymptote. Notice, that using our original *h(x),* the horizontal asymptote is found by dividing the slope of the linear function in the numerator by the slope of the linear function in the denominator. This follows directly from taking the limit of the function.

Horizontal asymptotes rules:

$$f(x) = \frac{Numerator}{Denominator} = \frac{N(x)}{D(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \ldots a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \ldots b_1x + b_0}$$

Let degree of numerator = n and degree of denominator = m*. Let* a_n *be the coefficient of the variable with the highest power in the numerator, and* b_m *be the coefficient of the variable with the highest power in the denominator.*

1) $n < m$, *the horizontal asymptote is y = 0.*
2) $n = m$, *the horizontal asymptote is y =* $\frac{a_n}{b_m}$.
3) $n > m$, *there is no horizontal asymptote.*

An important mathematical concept to remember is that a graph simply approaches an asymptote, but will never cross the asymptote.

Mathematical Focus 3: Graphs of Rational Functions

Any two linear expressions divided by one another will form a rational function that can be graphed in the coordinate plane. In the following section, we will discuss the necessary steps to graph a rational function correctly.

Start by finding the vertical and horizontal asymptotes. Then, graph and label the asymptotes in the coordinate plane. Use the asymptotes to determine the domain and range of the rational function. In order to find the domain of a rational function, set the denominator equal to zero (i.e. the vertical asymptote). We do this because our domain cannot contain any real numbers which would make the denominator zero (i.e. make the function undefined). Thus, the domain will be all real values except for the x – value where the denominator is equal to zero. In order to find the range of a rational function, we use the same concept but look at the horizontal asymptote(s) rather than the vertical asymptote(s).

Along with finding the domain and range, we must find the intercepts in order to graph a rational function. We find the y - intercept by setting x = 0 and solving for y, and we find the x – intercept by setting y = 0 and solving for x. The intercepts are also known as the zeros of the function and the x – intercept is more specifically known as the root(s). Once students have graphed the asymptotes, the x – intercept, and the y – intercept, they will then use a t – chart to find other coordinate points to graph the function. Then, students will analyze the graph and determine when the function is increasing or decreasing. Take the example shown below:

$$Let\ f(x) = 6x + 2, g(x) = 2x + 3, and\ h(x) = \frac{f(x)}{g(x)}.$$

$$h(x) = (6x + 2)/(2x + 3)$$

We can determine that the rate of change of dividing two linear functions is not constant with respect to x, thus not linear.

For this specific case, the result of dividing the two linear equations gives us a function of the ratios of the two linear equations. Thus this is a rational function and more specifically a hyperbola.

Observing the graph of the hyperbola we can notice that the hyperbola approaches, but never reaches, two lines. These lines are called asymptotes and be found using our two linear functions. The denominator of h(x) can tell us where h(x) doesn't exist, giving us our vertical asymptote. This is simply done by setting the denominator equal to zero since we know dividing by zero is undefined.

$$2x + 3 = 0$$

$$2x = -3$$

$$x = -3/2$$

To find the other asymptote, which will be horizontal, we solve for the inverse of h(x) and set it's denominator equal to zero as well. Allow h(x) to be represented as y to solve for the inverse.

$$y = (6x + 2)/(2x + 3)$$

$$y(2x + 3) = (6x + 2)$$

$$2xy + 3y = 6x + 2$$

$$2xy - 6x = 2 - 3y$$

$$x(2y - 6) = (2 - 3y)$$

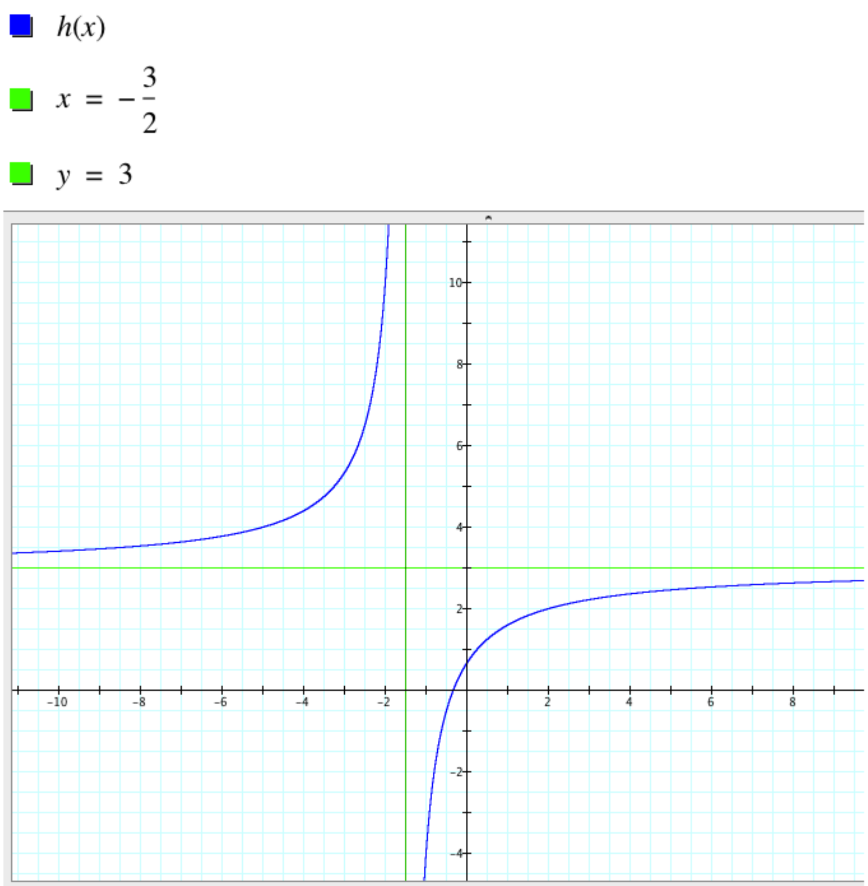
$$x = (2 - 3y)/(2y - 6)$$

Now that we've found the inverse, let's set it's denominator equal to zero to find our undefined value for x.

$$2y - 6 = 0$$

$$2y = 6$$

$$y = 3$$



Mathematical Focus 4: Transformations of Functions

Focus 4: Rational functions in the form $f(x) = \frac{ax+b}{cx+d}$ can be graphed by transforming the graph of $f(x) = \frac{1}{x}$ using four simple steps.

Recall from the long division algorithm located in Focus 1 that

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d}$$

Starting with the graph $f(x) = \frac{1}{x}$ we can take 4 steps to get to $f(x) = \frac{a}{c} + \frac{b-\frac{ad}{c}}{cx+d}$

Step 1. By multiplying $\frac{1}{x}$ by a scalar of $\frac{1}{c}$ a stretch/shrink will take place.

$$\frac{1}{x} \to \frac{1}{cx}$$

Step 2. By adding a factor of d to the denominator of our previous equation, a horizontal shift will be observed.

$$\frac{1}{cx} \to \frac{1}{cx+d}$$

Step 3. By multiplying our previous equation by $b - \frac{ad}{c}$ another stretch/shrink will occur.

$$\frac{1}{cx+d} \to \frac{b - \frac{ad}{c}}{cx+d}$$

Step 4. Finally, by adding $\frac{a}{c}$ to our previous equation, the desired function is obtained.

$$\frac{b - \frac{ad}{c}}{cx+d} \to \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d}$$