Situation: Dividing Binomials

June 21, 2013

<u>Prompt</u> : An Algebra II class has been examining the product of two linear expressions $(ax+b)(cx+d)$. Well into the class a student asks "What would happen if we DIVIDED one linear expression by another?"

<u>Commentary</u> We can think of dividing binomials in multiple ways: as a quotient function via polynomial long division; as the graph of a rational function with vertical and horizontal asymptotes; or, also via long division, the graph of (1) a horizontal line with a hole in it; or (2) a rectangular hyperbola arrived at by geometrically transforming - without changing the shape of - its parent function the rectangular hyperbola $y = \frac{1}{x}$. We will generally consider only these two major cases, and not the case where $c = 0$ and the resulting function is a scalar multiple of $f(x) = ax + b$, or the cases where $a = b = 0$, $c = d = 0$, or $a = b = c = d = 0$.

<u>Mathematical Foci</u>

<u>Focus 1</u> : *We can divide any two binomials (ax+b) & (cx+d) using the division algorithm for polynomials and classify the graph of the resulting functions based on criterion for a, b, c, & d.*

The Euclidean Division Algorithm states that for $p, q, \in \mathbb{R}$, $\exists\, d, r$ such that

$$p = qd + r$$

In short, two integers will always divide into a factor multiplied by the quotient, plus a remainder smaller than the quotient.
Similarly, the division algorithm for polynomials states that for $p(x)$, $q(x) \in P[x]$, $\exists\, d(x), r(x)$ such that

$$p(x) = d(x)q(x) + r(x),$$

where $\deg(r(x)) < \deg(q(x))$.
If $r \equiv 0$, then $q(x)$ is said to *divide* $p(x)$, denoted $q(x) \,|\, p(x)$, and thus $q(x)$ is a *factor* of $p(x)$.
In the case of two linear binomials, $q(x) = cx+d$ and $p(x) = ax+b$, and we have that $q(x) \,|\, p(x) \Leftrightarrow q(x) = cx + d = k(ax + b) = kp(x)$ on $\mathbb{R} \setminus \{-\frac{d}{c}\}$, where $k \in \mathbb{Q}$. In this case, in other words, the line $ax + b$ is a scalar multiple of $cx + d$, *but is a line with a hole in it* precisely when the x value is such that $cx + d = 0$ and the function $y = \frac{ax+b}{cx+d}$ is undefined.

If $p(x) \nmid q(x)$, we use the division algorithm.

For $y = \frac{ax+b}{cx+d}$, we calculate

$$\begin{array}{r|l} & \frac{a}{c} \\ \hline cx + d \,) & ax + b \\ & -\; ax \quad \frac{ad}{c} \\ \hline & \quad b - \frac{ad}{c} \end{array}$$

We thus get via long division

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{\frac{bc-ad}{c}}{cx+d}$$

Note that this function will be undefined at $x = -\frac{d}{c}$ $\forall c, d \in \mathbb{R}$ such that $c \neq 0$. (If $c = 0$, then our function becomes a line divided by a scalar and thus remains a line with no hole so long as $d \neq 0$.
($\bigstar$) Also note that the graph of $y = \frac{a}{c} + \frac{\frac{bc-ad}{c}}{cx+d}$ is a horizontal line if $bc - ad = 0$ and $c \neq 0$.
We therefore have a criterion for the graph of $y = \frac{ax+b}{cx+d}$. The graph is the horizontal line $y = \frac{a}{c}$ with a hole at $(-\frac{d}{c}, \frac{a}{c})$ if and only if $bc - ad = 0$.
With this information and fact $\bigstar$, we know that $y = \frac{ax+b}{cx+d}$ has a graph that is not a horizontal line with a hole in it if and only if
1) $c \neq 0$, and
2) $bc - ad \neq 0$.
Note that $bc - ad = 0 \Rightarrow ad - bc = 0$. Therefore, criteria (2) can be restated as
2) $det[A] \neq 0$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

<u>Focus 2</u> : $y = \frac{ax+b}{cx+d}$ *is a rational function with denominator of odd degree.*

In Focus 1, we discussed dividing polynomials. The class of all functions $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$, $q(x)$ are polynomials with integer coefficients are called $rational functions$.
With rational functions $f(x) = \frac{p(x)}{q(x)}$ such that $q(x) \nmid p(x)$, we have two cases:
1) $q(x)$ has even degree.
In this case, there need not exist $x_0 \in \mathbb{R}$ such that $q(x_0) = 0$.
Thus, $\frac{p(x)}{q(x)}$ may be well-defined $\forall\, x \in \mathbb{R}$.
For example, $g(x) = \frac{x}{x^2+2}$ is well-defined on $\mathbb{R}$.
2) $q(x)$ has odd degree.
If $q(x)$ has odd degree, then there necessarily exists $x_0 \in \mathbb{R}$ such that $q(x_0) = 0$, and thus $q(x)$ is not defined at x_0.
For $y = \frac{ax+b}{cx+d}$ where $(cx+d) \nmid (ax+b)$, $deg(cx+d) = 1$ and thus there exists
$x_0 = (-\frac{d}{c})$ such that $c(x_0) + d = -d + d = 0$.
Therefore, $y = \frac{ax+b}{cx+d}$ is a rational function with domain $\mathbb{R} \setminus \{-\frac{d}{c}\}$ and range $\{y = \frac{ax+b}{cx+d} \,|\, x \neq -\frac{d}{c}\}$, and one root at $x = -\frac{b}{a}$.

Focus 3 : *Any rational function* $f = \frac{p}{q}$ *such that* $q \nmid p$ *has vertical asymptotes at all points* a *such that* $q(a) = 0$.
A rational function $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote at $x = a$ if one of the following four cases is satisfied:
1) $\lim_{x \to a^+} f(x) = \infty$
2) $\lim_{x \to a^-} f(x) = \infty$
3) $\lim_{x \to a^+} f(x) = -\infty$
4) $\lim_{x \to a^-} f(x) = -\infty$

In other words, f has a vertical asymptote at $x = a$ if $f(x)$ approaches infinity or negative infinity as x approaches a from the right or left. It is worth noting that for $f = \frac{p}{q}$, a is precisely a value of q such that $q(a) = 0$. Thus, as $q(x) \to q(a) = 0$, $\lim_{x \to a^{\pm}} \to \pm\infty$.

On the other hand, a non-vertical asymptote of rational function $f(x)$ is either *horizontal* or *oblique*. A horizontal asymptote occurs at $y = b$ when $f(x) \to b$ as $x \to \infty$ and/or $x \to -\infty$. The value $y = b$ of a horizontal asymptote of a rational function is necessarily finite, and there can be at most one horizontal asymptote for any *rational function*.

The third and last kind of asymptote is oblique. An oblique asymptote of f is a non-constant line of the form $y = mx + b$ such that $\lim_{x \to \pm\infty}(f(x) - (mx + b)) = 0$.

For $f = \frac{p}{q}$, there are strict rules for determining the existence of horizontal or oblique asymptotes based on the degrees of p, q.

1) If $deg(p) - deg(q) = 1$, there will be an oblique asymptote at $y = mx + b$. One can calculate m & b using the polynomial long division discussed in **Focus 1**.
- In particular, $m = \frac{b}{a}$, where a, b are the leading coefficients of p, q, respectively.

2) If $deg(p) - deg(q) > 1$, f has no horizontal asymptote.
- In this case, f necessarily approaches $\pm\infty$ as $x \to \pm\infty$. The numerator dominates the denominator, and such a function is often called *top-heavy*.

3) If $deg(p) - deg(q) = 0$, then $deg(p) = deg(q) = m$, and there is a horizontal asymptote at $y = \frac{a}{b}$, where a, b are the leading coefficients of p, q, respectively.
- To see this, one can divide both p, q by x^m and let $x \to \pm\infty$. All terms of degree less than m will go to zero as $x \to \pm\infty$, and the only non-zero term that will remain is $\frac{a}{b}$.

4) If $deg(p) - deg(q) < 0$, then there is a horizontal asymptote at $y = 0$.
- Using the same method as in (3), one finds that f is reduced to a constant over a polynomial with degree greater than 1. The denominator thus dominates the numerator, and $f \to 0$ as $x \to \pm\infty$.

For $y = \frac{p(x)}{q(x)} = \frac{ax+b}{cx+d}$ where $q \nmid p$, we have a vertical asymptote at $x = -\frac{d}{c}$ and a horizontal asymptote at $y = \frac{a}{c}$ because p, q have the same degree.

Focus 4 : $y = \frac{1}{x}$ *is a rectangular hyperbola and the parent function of all rectangular hyperbolas,* *including $y = \frac{ax+b}{cx+d}$. The latter function can thus be arrived at by applying geometric transformations to $y = \frac{1}{x}$.*

In **Focus 1**, we used polynomial division to convert our binomial quotient

$$\frac{ax+b}{cx+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cx+d}$$

We need at most three geometric transformations to convert $y = \frac{1}{x}$ into $y = \frac{a}{c} + \frac{b-\frac{ad}{c}}{cx+d} = \frac{\frac{bc-ad}{c^2}}{x-(-\frac{d}{c})} + \frac{a}{c}$.

1) Horizontal Translation: $y = \frac{1}{x-h}$ will *horizontally translate* the hyperbola $|h|$ units to the left if $h < 0$ and h units to the right if $h > 0$.

2) Vertical Translation: $y = \frac{1}{x-h} + k$ will *vertically translate* the hyperbola k units up if $k > 0$ and $|k|$ units down if $k < 0$.

3) Dilation: $y = \frac{r}{x-h} + k$ will *dilate* the hyperbola by a factor of r, vertically stretching or contracting the hyperbola by a factor of $r \in \mathbb{Q}$ for $r > 0$.

4) Reflection: $y = \frac{r}{x-h} + k$ will dilate as well as *reflect* the hyperbola over the line $x = -\frac{d}{c}$ if $r \in \mathbb{Q}$ is such that $r < 0$.

Therefore, our quotient $y = \frac{ax+b}{cx+d}$ is a rectangular hyperbola with a graph translated left or right by $|-\frac{d}{c}|$ units; translated vertically up or down by $|\frac{a}{c}|$ units; and is dilated by a factor of $|\frac{bc-ad}{c^2}|$. Since $deg(ax+b) = 1 = deg(cx+d)$, we have that our quotient has a vertical asymptote at $x = -\frac{d}{c}$ and a horizontal asymptote at $y = \frac{a}{c}$. Thus, we can see that a, c determine the horizontal asymptote. If their signs agree, the asymptote will be positive; similarly, the more positive or negative of slope our numerator line has, the more positive or negative will be our horizontal asymptote y. See Figures 1 & 2.

In Figure 2, we see that our dilation factor $r = -\frac{1}{9} < 0$, and thus the hyperbola in Figure 1 is reflected over the line $x = -\frac{d}{c}$ in Figure 2.

Post Commentary: In our discussion of the division of two binomials, we did not consider the inverse of the rational function at hand. Here, will look at this aspect of the rational function. When we divide two linear binomials, we end up with $f(x) = \frac{ax+b}{cx+d}$, a one-to-one rational function. We can define a function $f^{-1}(x)$ called the inverse of f which reverses the correspondence defined by $f(x)$. To find the inverse of f, we interchange x and y in the equation and solve the equation for y in terms of x.When we apply this sequence of steps to $f(x) = \frac{ax+b}{cx+d}$, we obtain the following:

$$f^{-1} = \frac{dx-b}{-cx+a}$$

When we compare the graphs of f and f^{-1}, we note that f^{-1} is a reflection of $f(x)$ about the line

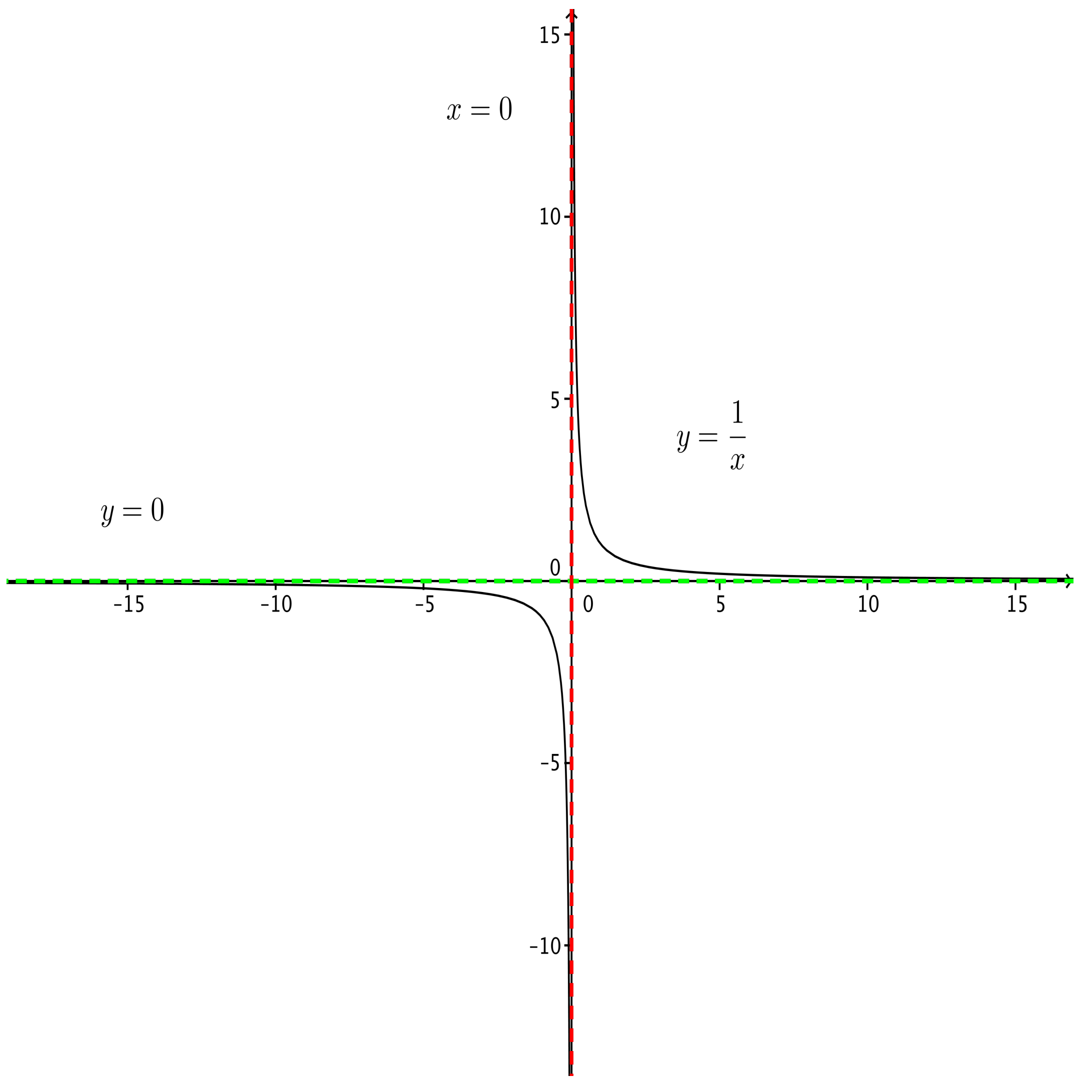

Figure 1: Graph of a basic rectangular hyperbola with vertical asymptote at $x = 0$ and horizontal asymptote at $y = 0$.

$y = x$. This is to be expected, since the inverse of a function "undoes" the correspondence defined by $f(x)$.
Additionally, for $f(x)$ we calculated the vertical asymptote, $x = -\frac{d}{c}$ and the horizontal asymptote $y = \frac{a}{c}$. By using the same method of calculation for f^{-1} that we used for f, we find the vertical asymptote $x = \frac{a}{c}$ and the horizontal asymptote $y = -\frac{d}{c}$ of f^{-1}. We notice that the value for the vertical asymptote of f is equivalent to the value of the horizontal asymptote of f^1. Similarly, the value of the horizontal asymptote of f is the value for the vertical asymptote of f^1.
This is in alignment with our geometric representation of $f^1(x)$ as a reflection of $f(x)$ about the line $y = x$. In other words, because the asymptotes for $f(x) = \frac{ax+b}{cx+d}$ are horizontal and vertical (i.e. non-oblique), the asymptotes are perfectly flipped from vertical to horizontal and vice-versa.

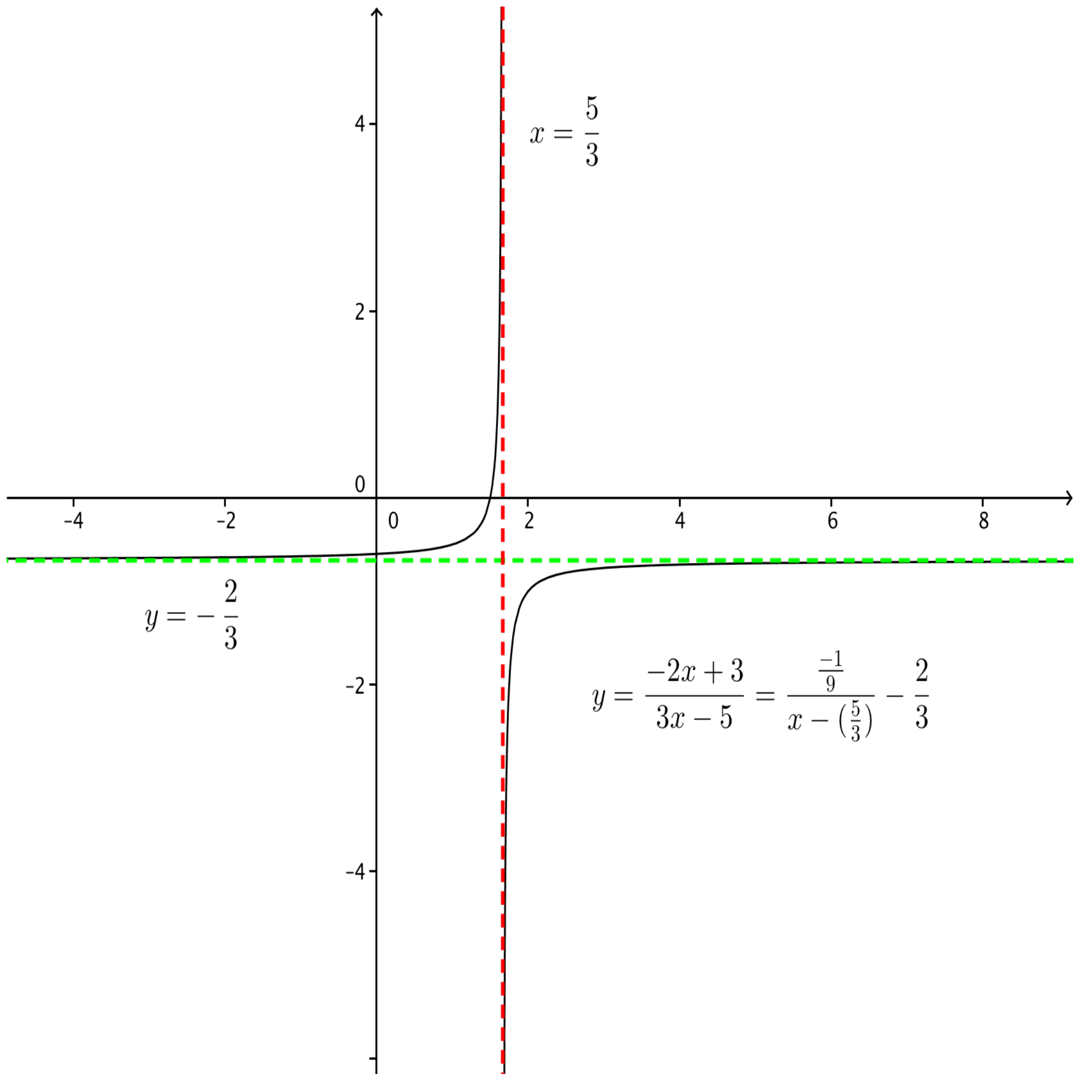

Figure 2: Graph of $y = \frac{ax+b}{cx+d}$.