Georgia Performance Standards Framework

Mathematics 1 Unit 2
Algebra Investigations

INTRODUCTION: In this unit students develop skill in adding, subtracting, multiplying, and dividing elementary polynomial, rational, and radical expressions. By using algebraic expressions to represent quantities in context, students understand algebraic rules as general statements about operations on real numbers. The focus of the unit is the development of students' abilities to read and write the symbolically intensive language of algebra.

ENDURING UNDERSTANDINGS:

- Algebraic equations can be identities that express properties of operations on real numbers.
- Equivalence of algebraic expressions means that the expressions have the same numerical value for all possible values of the variable.
- Equivalent expressions are useful tools in computation and problem solving.
- It takes only one counterexample to show that a general statement is not true.

KEY STANDARDS ADDRESSED:

MM1A2. Students will simplify and operate with radical expressions, polynomials, and rational expressions.

a. Simplify algebraic and numeric expressions involving square root.
b. Perform operations with square roots.
c. Add, subtract, multiply, and divide polynomials.
d. Expand binomials using the Binomial Theorem
e. Add, subtract, multiply, and divide rational expressions.
f. Factor expressions by greatest common factor, grouping, trial and error, and special products limited to the formulas below.

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)(x-y) = x^2 - y^2$$

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

g. Use area and volume models for polynomial arithmetic.

Student Edition

MM1A3. Students will solve simple equations.

a. Solve quadratic equations in the form $ax^2 + bx + c = 0$, where $a = 1$, by using factorization and finding square roots where applicable.

* *In this unit, use of the binomial theorem limits n to two or three, in future units, the value of n will be increased.*

$$(a \pm b)^n = a^n \pm {}_nC_{n-1}\, a^{n-1}b + {}_nC_{n-2}\, a^{n-2}b^2 \pm \ldots\ldots + {}_nC_1\, ab^{n-1} \pm b^n$$

<u>RELATED STANDARDS ADDRESSED:</u>

MM1G2. Students will understand and use the language of mathematical argument and justification.

a. Use conjecture, inductive reasoning, deductive reasoning, counterexamples, and indirect proof as appropriate

MM1P1. Students will solve problems (using appropriate technology).

a. Build new mathematical knowledge through problem solving.
b. Solve problems that arise in mathematics and in other contexts.
c. Apply and adapt a variety of appropriate strategies to solve problems.
d. Monitor and reflect on the process of mathematical problem solving.

MM1P2. Students will reason and evaluate mathematical arguments.

a. Recognize reasoning and proof as fundamental aspects of mathematics.
b. Make and investigate mathematical conjectures.
c. Develop and evaluate mathematical arguments and proofs.
d. Select and use various types of reasoning and methods of proof.

MM1P3. Students will communicate mathematically.

a. Organize and consolidate their mathematical thinking through communication.
b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
c. Analyze and evaluate the mathematical thinking and strategies of others.
d. Use the language of mathematics to express mathematical ideas precisely.

MM1P4. Students will make connections among mathematical ideas and to other disciplines.

a. Recognize and use connections among mathematical ideas.
b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

MM1P5. Students will represent mathematics in multiple ways.

a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.

UNIT OVERVIEW

Prior to this unit, students need to have worked extensively with operations on integers, rational numbers, and square roots of nonnegative integers as indicated in the Grade 6 – 8 standards for Number and Operations.

In the unit students will apply and extend all of the Grade 6 – 8 standards related to writing algebraic expressions, performing operations with algebraic expressions, and working with relationships between variable quantities. Students are assumed to have a deep understanding of linear relationships between variable quantities. Students should understand how to find the area of triangles, rectangles, squares, and trapezoids and know the Pythagorean theorem.

In the unit students will apply the basic function concepts of domain, range, rule of correspondence, and interpreting graphs of functions learned in Unit 1 of Mathematics 1.

The unit begins with intensive work in writing linear and quadratic expressions to represent quantities in a real-world context. The initial focus is developing students' abilities to read and write meaningful statements using the language of algebra. In the study of operations on polynomials, the special products of standard MM1A2 are studied as product formulas and interpreted extensively through area models of multiplication. Factoring polynomials and solving quadratic equations by factoring are covered in a later unit, but the work with products and introduction of the zero factor property are designed to build a solid foundation for these later topics.

The work with rational and radical expressions is grounded in work with real-world situations that show applications of working with such expressions. Students need to practice computational skills, but extensive applications are needed to demonstrate that the skills have important applicability in modeling and understanding the world around us. The work with rational expressions includes consideration of the application to calculating average speeds.

Students are given many opportunities to use geometric reasoning to justify algebraic equivalence and understand algebraic rules as statements about real number operations. Early in their study of algebra, students may have difficulty grasping the full content of abstract algebraic statements. Many of the questions in the tasks of this unit are intended to guide students to see how a geometric or other relationship from a physical context is represented by an algebraic expression. So, although some of the questions may seem very simple, it is important that they not be skipped. Gaining the ability to see all the information in an abstract algebraic statement takes time and lots of practice.

Algebra is the language that allows us to make general statements about the behavior of the numbers and gaining facility with this language is essential for every educated citizen of the twenty-first century. Throughout this unit, it is important to:

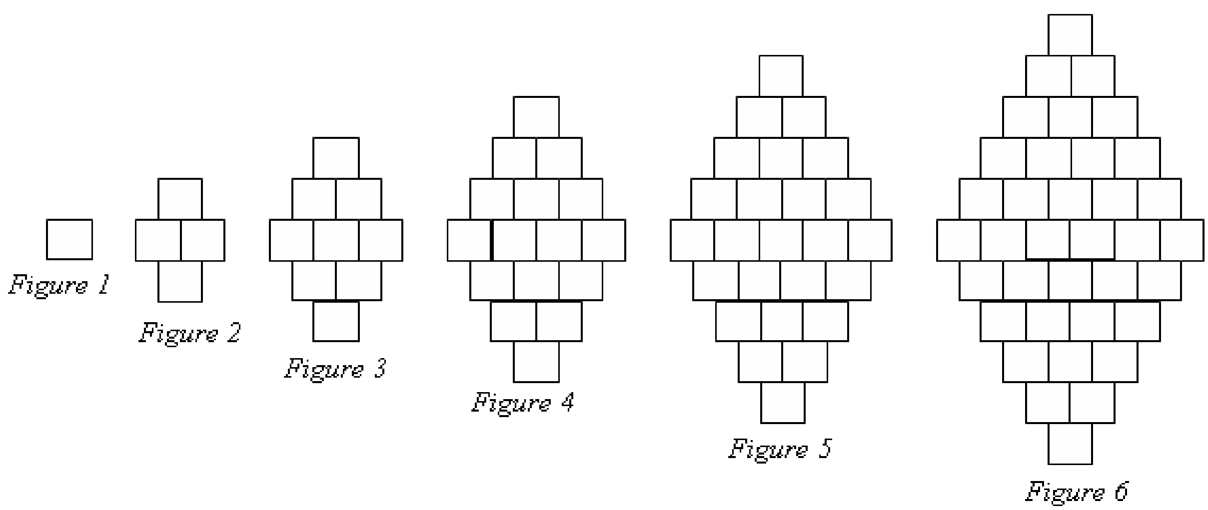
- Require students to explain how their algebraic expressions, formulas, and equations represent the geometric or other physical situation with which they are working.
- Encourage students to come up with many different algebraic expressions for the same quantity, to use tables and graphs to verify expressions are equivalent, and to use algebraic properties to verify algebraic equivalence.
- Make conjectures about relationships between operations on real numbers and then give geometric and algebraic explanations of why the relationship always holds or use a counterexample to show that the conjecture is false.

TASKS: The remaining content of this framework consists of student tasks or activities. The first is intended to launch the unit. Activities designed to allow students to build their own algebraic understanding through exploration follow. The last task is designed to demonstrate the type of assessment activities students should be comfortable with by the end of the unit. Thorough Teacher's Guides which provide solutions, discuss teaching strategy, and give additional mathematical background are available to accompany each task.

Tiling Learning Task

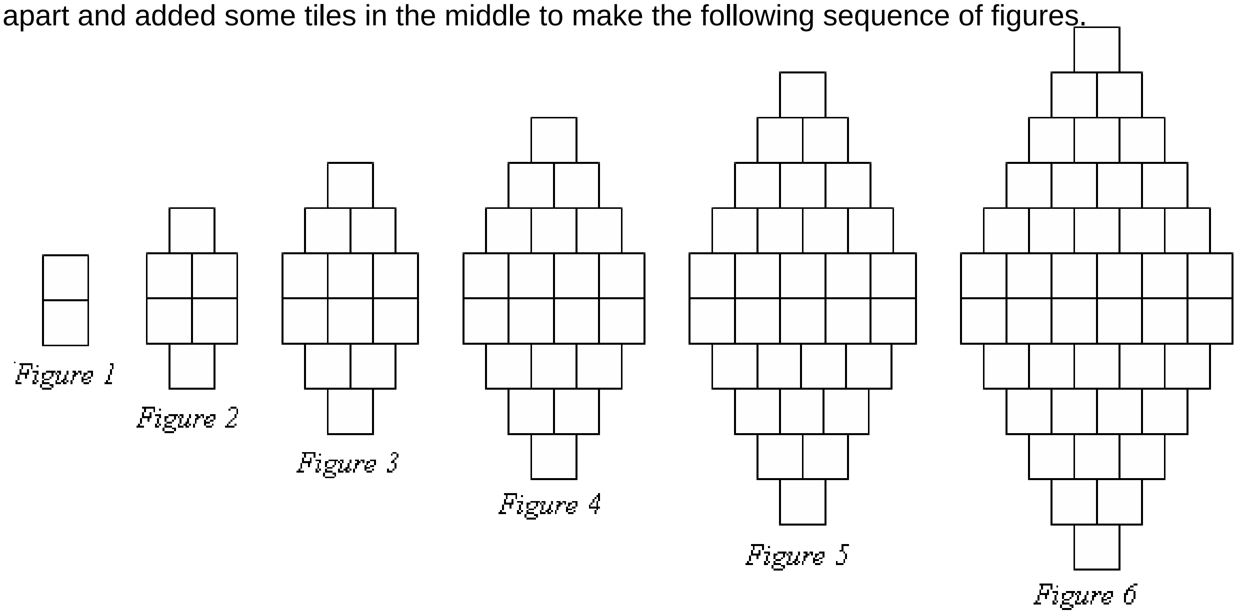
Latasha and Mario are high school juniors who worked as counselors at a day camp last summer. One of the art projects for the campers involved making designs from colored one-square-inch tiles. As the students worked enthusiastically making their designs, Mario noticed one student making a diamond-shaped design and wondered how big a design, with the same pattern, that could be made if all 5000 tiles available were used. Later in the afternoon, as he and Latasha were putting away materials after the children had left, he mentioned the idea to Latasha. She replied that she saw an interesting design too and wondered if he were talking about the same design. At this point, they stopped cleaning up and got out the tiles to show each other the designs they had in mind.

Mario presented the design that interested him as a sequence of figures as follows:



1. To make sure that you understand the design that was of interest to Mario, answer the following questions.
    a) How many rows of tiles are in each of Mario's figures?
    b) What pattern do you observe that relates the number of rows to the figure number? Explain in a sentence.
    c) Use this pattern to predict the number rows in Figure 12, Figure 47,and Figure 111 if these figures were to be drawn.
    d) Write an algebraic expression for the number of rows in Figure k. Explain why your pattern will always give the correct number of rows in Figure k. Can your expression be simplified? If so, simplify it.
    e) What is the total number of tiles in each figure above?
    f) What pattern do you observe that relates the total number of tiles to the figure number? Explain in a sentence.
    g) Use this pattern to predict the total number of tiles in Figure 12, Figure 47, and Figure 111 if these figures were to be drawn.
    h) Write an algebraic expression for the total number of tiles in Figure k. Explain why your pattern will always give the correct total number of tiles in Figure k.

When Latasha saw Mario's figures, she realized that the pattern Mario had in mind was very similar to the one that caught her eye, but not quite the same. Latasha pushed each of Mario's designs apart and added some tiles in the middle to make the following sequence of figures.



2. Answer the following questions for Latasha's figures.
   a) How many rows of tiles are in each of the figures above?
   b) What pattern do you observe that relates the number of rows to the figure number? Explain in a sentence.
   c) Use this pattern to predict the number rows in Figure 12, Figure 47, and Figure 111 if these figures were to be drawn.
   d) Write an algebraic expression for the number of rows in Figure k. Explain why your pattern will always give the correct number of rows in Figure k. Can your expression be simplified? If so, simplify it.
   e) What is the total number of tiles in each figure above?
   f) What pattern do you observe that relates the total number of tiles to the figure number? Explain in a sentence.
   g) Use this pattern to predict the total number of tiles in Figure 12, Figure 47, and Figure 111 if these figures were to be drawn.
   h) Write an algebraic expression for the total number of tiles in Figure k. Explain why your pattern will always give the correct total number of tiles in Figure k.
   i) Give a geometric reason why the number of tiles in Figure k is always an even number. Look at the algebraic expression you wrote in part d.
   j) Give an algebraic explanation of why this expression always gives an even number.

[Hint: If your expression is not a product, use the distributive property to rewrite it as a product.]

3. Mario started the discussion with Latasha wondering whether he could make a version of the diamond pattern that interested him that would use all 5000 tiles that they had in the art supplies. What do you think? Explain your answer. If you can use all 5000 tiles, how many rows will the design have? If a similar design cannot be made, what is the largest design that can be made with the 5000 tiles, that is, how many rows will this design have and how many tiles will be used?

4. What is the largest design in the pattern Latasha liked that can be made with no more than 5000 tiles? How many rows does it have? Does it use all 5000 tiles? Justify your answers.

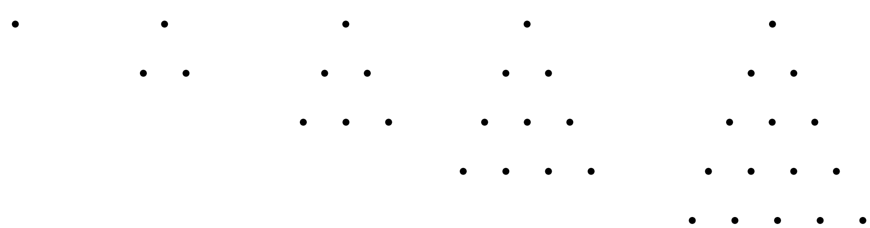
Let M_1, M_2, M_3, M_4, and so forth represent the sequence of numbers that give the total number of tiles in Mario's sequence of figures.

Let L_1, L_2, L_3, L_4 and so forth represent the sequence of numbers that give the total number of tiles in Latasha's sequence of figures.

5. Write an equation that expresses each of the following:
   a) the relationship between L_1 and M_1
   b) the relationship between L_2 and M_2
   c) the relationship between L_3 and M_3
   d) the relationship between L_4 and M_4
   e) the general relationship between L_k and M_k, where k can represent any positive integer.

Triangular numbers are positive integers such that the given number of dots can be arranged in an equilateral triangle. The first few triangular numbers are as follows.

1 3 6 10 15



Let T_1, T_2, T_3, T_4, and so forth represent the sequence of triangular numbers.

6. a) Examine the arrangement of dots for T_4. How many dots are in row 1? row 2? row 3? row 4?
   b) Write T_4 as a sum of four positive integers.
   c) Write each of T_2, T_3, and T_5 as a sum 2,3, and 5 positive integers, respectively.
   d) Explain what sum you would need to compute to find T_{12}. Compute T_{12} doing the addition yourself.
   e) Explain what sum you would need to compute to find T_{47}. Use technology to find T_{47}.

7. The triangular numbers could also be represented by arrangements of square tiles instead of dots.
   a) Draw the first few figures to represent triangular numbers using square tiles.
   b) Compare these figures to Latasha's diamond-shaped figures. Write a sentence comparing the triangular-number figures and Latasha's figures.
   c) Write an equation using L_k and T_k to express the relationship between the number tiles in the *k*-th one of Latasha's figures and the *k*-th triangular number figure.
   d) Check your equation by comparing the numbers L_1 and T_1, L_2 and T_2, L_3 and T_3, L_4 and T_4, L_5 and T_5, L_{12} and T_{12}, and L_{47} and T_{47}
   e) Write a formula to calculate T_k, the k-th triangular number, without summing the first *k* positive integers.
   f) Check your formula by using it to calculate T_1, T_2, T_3, T_4, T_5, T_{12}, and T_{47}.
   g) Give a geometric reason why your formula always gives a whole number.
   h) Give an algebraic reason why the formula must always give a whole number.
   i) What sum does your formula calculate?
8. a) Compare the triangular-number figures to Mario's figures. Write a sentence comparing the triangular-number figures and Mario's figures.
   b) Write an equation using M_k and T_k to express the relationship between the number tiles in the *k*-th one of Mario's figures and the *k*-th triangular number figure.
   c) What is another name for the sequence M_1, M_2, M_3, M_4, . . . ?
   d) Write a sentence expressing the relationship expressed in part b); use the familiar name for the numbers in the sequence M_1, M_2, M_3, M_4, ...

Tiling Pools Learning Task

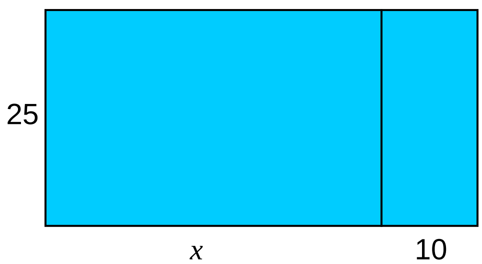
In this task, you will continue to explore how different ways of reasoning about a situation can lead to algebraic expressions that are different but equivalent to each other. We will use swimming pools as the context throughout this task.

In the figures below there are diagrams of swimming pools that have been divided into two sections. Swimming pools are often divided so that different sections are used for different purposes such as swimming laps, diving, area for small children, etc.

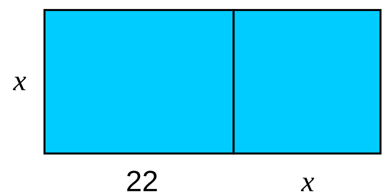
(a) For each pool, write two different but equivalent expressions for the total area.

(b) Explain how these diagrams and expressions illustrate the Distributive Property.

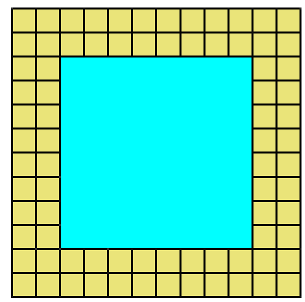
1.



2.



In-ground pools are usually surrounded by a waterproof surface such as concrete. Many homeowners have tile borders installed around the outside edges of their pools to make their pool area more attractive. Superior Pools specializes in custom pools for residential customers and often gets orders for square pools of different sizes. The diagram at the right shows a pool that is 8 feet on each side and is surrounded by two rows of square tiles. Superior Pools uses square tiles that are one foot on each side for all of its tile borders.

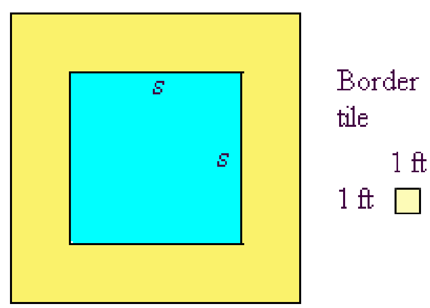


The manager at Superior Pools is responsible for telling the installation crew how many border tiles they need for each job and needs an equation for calculating the number of tiles needed for a square pool depending on the size of the pool. Let N represent the total number of tiles needed when the length of a side of the square pool is s feet and the border is two tiles wide.

3. Write a formula in terms of the variable s that can be used to calculate N.

4. Write a different but equivalent formula that can be used to calculate N.

5. Give a geometric explanation of why the two different expressions in your formulas for the number of border tiles are equivalent expressions. Include diagrams.

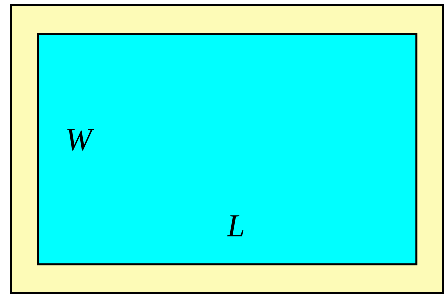


6. Use the Commutative, Associative, and/or Distributive properties to show that your expressions for the number of border tiles are equivalent.

* Adapted from the "Equivalent Expressions" section of *Say It With Symbols: Making Sense of Symbols* in the *Connected Mathematics 2* series.

Georgia Performance Standards Framework for Mathematics 1

Some customers who have pools installed by Superior Pools want larger pools and choose a rectangular shape that is not a square. Many of these customers also choose to have tile borders that are 2 tiles wide.

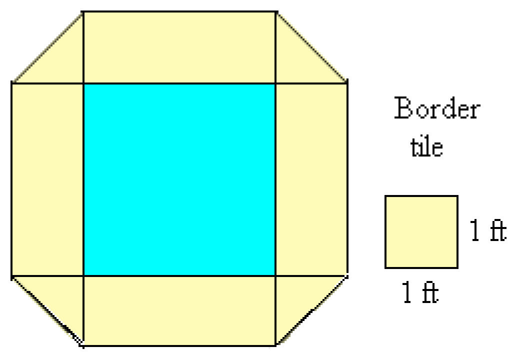


7. How many 1-foot square border tiles are needed to put a two-tile-wide border around a pool that is 12 feet wide and 30 feet long?

8. Write an equation for finding the number N of border tiles needed to put a two-tile-wide border around a pool that is L feet long and W feet wide. Explain, with diagrams, how you found your expression.

9. Explain why the area A of the tile border (in square feet) should equal the number of tiles that are needed for the border. Write an equation for finding the area A of the tile border using an expression that is different from but equivalent to the expression used in the equation for N given in answering question 8. Use algebraic properties to show that your expressions for A and N are equivalent.

A company that sells hot tubs creates a tile border for its products by placing 1-foot-square tiles along the edges of the tub and triangular tiles at the corners as shown. The triangular tiles are made by cutting the square tiles in half along a diagonal.



10. Suppose a hot tub has sides of length 6 feet. How many square tiles are needed for the border?

11. Write an equation for the number of square tiles N needed to create such a border on a hot tub that has sides that are s feet long.

12. Write a different but equivalent expression for the number of border tiles N. Explain why this expression is equivalent to the one given in your answer to question 11.

13. Below are three expressions that some students wrote for the number of tiles needed for the border of a square hot tub with sides *s* feet long.

(i) $4\left(\frac{s+s+1}{2}\right)$ (ii) $4\left(\frac{s}{2}+\frac{s}{4}\right)+2$ (iii) $(s+2)^2-4\left(\frac{1}{2}\right)-s^2$

(a) Use each expression to find the number of border tiles if $s = 0$.

(b) Do you think that the expressions are equivalent? Explain.

(c) Use each expression to find the number of tiles if $s = 10$. Does this result agree with your answer to part (b)? Explain.

(d) What can you say about testing specific values as a method for determining whether two different expressions are equivalent?

(e) Use algebraic properties to show the equivalence of those expressions in 11, 12, and 13 which are equivalent.

I've Got Your Number Learning Task

Equivalent algebraic expressions, also called algebraic identities, give us a way to express results with numbers that always work a certain way. In this task you will explore several "number tricks" that work because of basic algebra rules. It is recommended that you do this task with a partner.

Think of a number and call this number x.

Now think of two other numbers, one that is 2 more than your original number and a second that is 3 more than your original number. No matter what your choice of original number, these two additional numbers are represented by $x + 2$ and $x + 3$.

Multiply your $x + 2$ by $x + 3$ and record it as **Answer 1** __________ .

Find the square of your original number, x^2 ______ , five times your original number, $5x$ ______,

and the sum of these two numbers plus six, $x^2 + 5x + 6$, and record it as **Answer 2** __________.

Compare Answers 1 and 2. Are they the same number? They should be. If they are not, look for a mistake in your calculations.

Question: How did the writer of this task know that your Answers 1 and 2 should be the same even though the writer had no way of knowing what number you would choose for x?
Answer: Algebra proves it has to be this way.

To get Answer 1, we multiply the number $x + 2$ by the number $x + 3$: $(x + 2)(x + 3)$

We can use the distributive property several times to write a different but equivalent expression.

First treat (x + 2) as a single number but think of $x + 3$ as the sum of the numbers x and 3, and apply the distributive property to obtain:	$(x + 2)\cdot x + (x + 2)\cdot 3$
Now, change your point of view and think of $x + 2$ as the sum of the numbers x and 2, and apply the distributive property to each of the expressions containing $x + 2$ as a factor to obtain:	$x \cdot x + 2\cdot x + x\cdot 3 + 2\cdot 3$
Using our agreements about algebra notation, rewrite as:	$x^2 + 2x + 3x + 6$
Add the like terms $2x$ and $3x$ by using the distributive property in the other direction:	$2x + 3x = (2 + 3)x = 5x$
The final expression equivalent to $(x + 2)(x + 3)$ is:	$x^2 + 5x + 6$

The last expression is Answer 2, so we have shown that $(x + 2)(x + 3) = x^2 + 5x + 6$ no matter what number you choose for x. Notice that 5 is the sum of 2 and 3 and 6 is the product of 2 and 3.

These calculations are just an example of the following algebraic identity.

$$(x+a)(x+b)=x^2+(a+b)x+ab \qquad \text{(Identity 1)}$$

For the remainder of this task, we'll refer to the above equivalence as Identity 1. Note that we used $a = 2$ and $b = 3$ as our example, but your task right now is to show, geometrically, that a could represent any real number and so could b.

1. (a) Each of the diagrams below illustrates Identity 1. Match each diagram with one of the following cases for a and b.

Case 1: a positive, b positive — Figure _____

Case 2: a positive, b negative — Figure _____

Case 3: a negative, b positive — Figure _____

Case 4: a negative, b negative — Figure _____

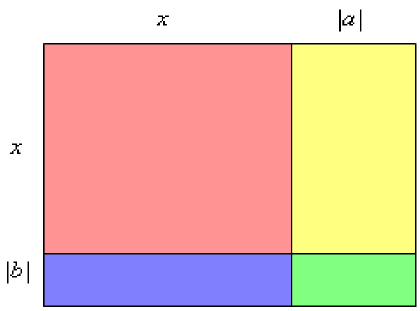


Figure A

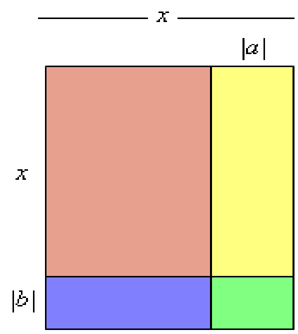


Figure B

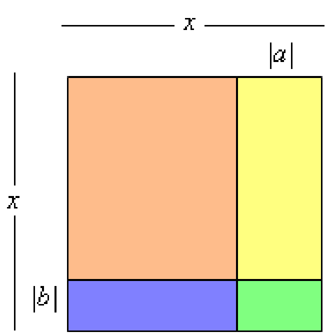


Figure C

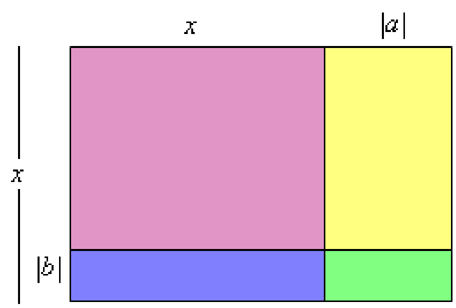


Figure D

(b) Thinking of the case it represents, for each diagram above, find the rectangle whose area is $(x+a)(x+b)$ and use a pencil to put diagonal stripes on this rectangle. Then explain how the diagram illustrates the identity. Note that when a is a negative number, $a=-|a|$, and when a is a positive number, then $a=|a|$; and similarly for b. Tell two of your explanations to another student and let that student explain the other two to you.

(c) What happens to the identity and the diagrams if a is 0? if b is 0? if both a and b are 0? Share explanations with another student.

2. Identity 1 can be used to give an alternate way to multiply two digit numbers that have the same digit in the ten's place.
For example, (31)(37) can be thought of as (30 + 1)(30 + 7) so we let x = 30, $a = 1$, and $b = 7$. Since $30^2 = 900$, 1 + 7 = 8 and $1 \cdot 7 = 7$,

$$(31)(37) = (30+1)(30+7) = 900 + (8)(30) + 7 = 900 + 240 + 7 = 1147.$$

Use Identity 1 to calculate each of the following products.

(a) (52)(57)

(b) (16)(13)

(c) (48)(42)

(d) (72)(75)

3. Look at your result for 2(c). You can get the answer another way by doing the following. Take 4, the common ten's digit, and multiply it by the next integer, 5, to get 20 as the number of hundreds in the answer. Then multiply the unit's digits 4 x 6 to get 24 for the last two digits. T

$$\begin{array}{r} 4\ 8 \\ \times\quad 4\ 2 \\ \hline 2\ 0\ \ 1\ 6 \end{array}$$

$4 \cdot 5 = \mathbf{20}$ $\qquad$ $8 \cdot 2 = \mathbf{16}$

Use this scheme to calculate the products below and verify the answers using Identity 1.

(a) (34)(36)

(b) (63)(67)

(c) (81)(89)

(d) (95)(95)

(e) In each of the products immediately above, look at the pairs of unit digits: 8 and 2 in the example, 4 and 6 for part (a), 3 and 7 in part (b), 1 and 9 in part (c), and 5 and 5 in part (d). What sum do each of the pairs have?_____

(f) Now let's use Identity 1 to see why this scheme works for these products with the property you noted in part (e). To represent two-digit numbers with the same ten's digit, start by using n to represent the ten's digit. So, n is 4 for the example and 3 for part (a). What is n for parts (b), (c), and (d)?

(g) Next, represent the first two-digit number as $10\,n + a$ and the second one as $10\,n + b$. In part (a):

$$(32)(38) = (30 + 2)(30 + 8) = (10n + a)(10n + b) \text{ for } n = 3,\ a = 2, \text{ and } b = 8.$$

List n _____, a ______, and b ______ for part (b).

List n _____, a ______, and b ______ for part (c).

List n _____, a ______, and b ______ for part (d)

(h) Use Identity 1 to multiply $(10\,n + a)(10\,n + b)$ where we have numbers like the example and parts (a), (b), (c),and (d). We are assuming that each of n, a, and b is a single digit number. What are we assuming about the sum $a + b$? $a + b =$ _______

$(10\,n + a)(10\,n + b) =$

Use the assumption about $a + b$ to write your result in the form $100\,k + ab$ where k is an expression containing the variable n and numbers.

Write the expression for k. ______________

Explain why k represents the product of two consecutive integers.

(j) Create three other multiplication exercises that can be done with this scheme and exchange your exercises with a classmate. When you both are done, check each other's answers.

(i)

(ii)

(iii)

(j) Explain the quick way to calculate each of the following:

(15)(15)

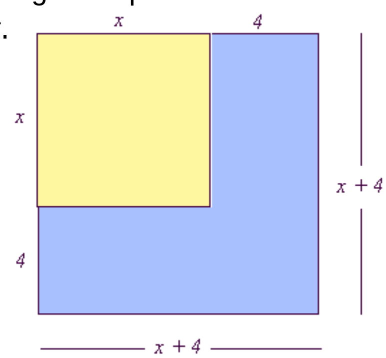
(45)(45)

(85)(85)

Why do each of these products fit the pattern of this question?

(k) Summarize in your own words what you've learned in the parts of question 3.

4. The parts of question 3 explored one special case of Identity 1. Now, let's consider another special case to see what happens when a and b in Identity 1 are the same number. Start by considering the square below created by adding 4 to the length of each side of a square with side length x.



(a) What is the area of the square with side length x?

(b) The square with side length $x + 4$ has greater area. Use Identity 1 to calculate its total area. When you use Identity 1, what are a, b, $a + b$?

(c) How much greater is the area of the square with side length $x + 4$? Use the figure to show this additional area. Where is the square with area 16 square units?

(d) How would your answers to parts (b) and (c) change if the larger square had been created to have side length $x + y$, that is, if both a and b are the same? If both a and b are both the same number y?

(e) At the right, draw a figure to illustrate the area of a square with side length $x + y$ assuming that x and y are positive numbers. Use your figure to explain the identity below.

$(x+y)^2 = x^2 + 2xy + y^2$ (Identity 2 - the Square of a Sum)

5. This identity gives a rule for squaring a sum. Use it to calculate each of the following by making convenient choices for x and y.

   (a) 302^2

   (b) 54^2

   (c) 65^2

   (d) 2.1^2

   (e) Look back at question 3, (k). Why will the rule for squaring a sum also work on those exercises? Can the method of Question 3 be used for the square of any sum?

6. We can extend the ideas of questions 4 and 5 to cubes.

   (a) What is the volume of a cube with side length 4?_______

   (b) What is the volume of a cube with side length x?_______

   (c) Now determine the volume of a cube with side length $x + 4$. First, use the rule for squaring a sum to find the area of the base of the cube. ______________________________

   Now use the distributive property several times to multiply the area of the base by the height, $x + 4$. Simplify your answer. ______________________________________

   (d) Repeat parts (b) and (c) for a cube with side length $x + y$. Write your result as a rule for the cube of a sum.

   area of the base of the cube ____________________________

   area of base multiplied by the height, $x + y$**:**_________________________________

(Identity 3 – the Cube of a Sum)

(e) Making convenient choices for x and y, use Identity 3 to find the following cubes.

11^3

23^3

101^3

Use the rule for cubing a sum to cube 2 = 1 + 1. Do you get the same number as (2)(2)(2)?

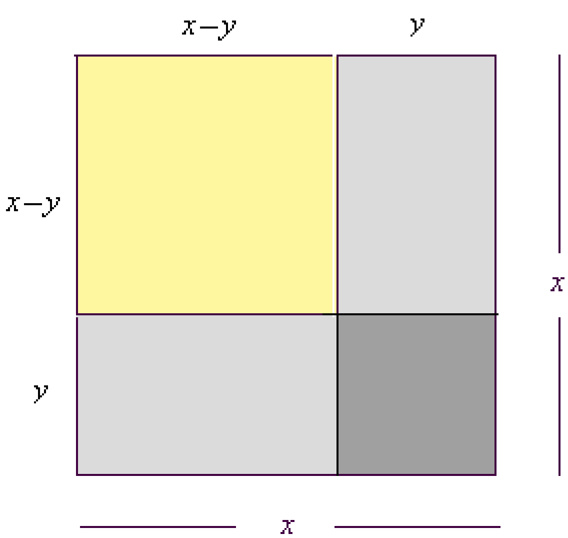
(f) Use the cube of the sum identity to simplify the following expressions.

$(t + 5)^3$

$(w + 2)^3$

7. (a) Let y represent any positive number. Substitute $-y$ for a and for b in Identity 1 to get the following rule for squaring a difference.

$$(x-y)^2 = x^2 - 2xy + y^2$$ (Identity 4 - The Square of a Difference)



(b) In the diagram find the square with side x, the square with side y, and two different rectangles with area xy. Now, use the diagram to give a geometric explanation of the rule for the Square of a Difference.

(c) By making a convenient choices for x and y, use the Square of a Difference identity to find the following squares. Note that 99 = 100 – 1, 38 = 40 – 2, and 17 = 20 – 3.

99^2

38^2

17^2

8. (a) Find a rule for the cube of a difference.

$(x-y)^3 =$ ________________________ (Identity 5 – The Cube of a Difference)

(b) Check your rule for the Cube of a Difference by using it to calculate: the cube of 1 using 1 = 2 – 1 and the cube of 2 using 2 = 5 – 3.

$1^3 = (2 - 1)^3$

$2^3 = (5 - 3)^3$

9. Now let's consider what happens in Identity 1 if a and b are opposite real numbers. Use Identity 1 to calculate each of the following. Substitute other variables for x as necessary.

(a) Calculate $(x + 8)(x - 8)$. Remember that $x - 8$ can also be expressed as $x + (-8)$.

(b) Calculate $(x - 6)(x + 6)$. Remember that $x - 6$ can also be expressed as $x + (-6)$.

(c) Calculate $(z + 12)(z - 12)$. Remember that $z - 12$ can also be expressed as $z + (-12)$.

(d) Calculate $(w + 3)(w - 3)$. Remember that $w - 3$ can also be expressed as $w + (-3)$.

(e) Calculate $(t - 7)(t + 7)$. Remember that $t - 7$ can also be expressed as $t + (-7)$.

(f) Substitute y for a and $-y$ for b in Identity 1 to find an identity for the product $(x + y)(x - y)$.

$(x + y)(x - y)$ ______________ (Identity 6)

10. Make appropriate choices for x and y to use Identity 6 to calculate each of the following.

(a) (101)(99)

(b) (22)(18)

(c) (45)(35)

(d) (63)(57)

(e) (6.3)(5.7)

(f) $\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)$

11. (a) In Question 10, you computed several products of the form $(x + y)(x - y)$ verifying that the product is always of the form $x^2 - y^2$. Thus, if we choose values for x and y so that $x = y$, then the product $(x + y)(x - y)$ will equal 0. If $x = y$, what is $x - y$?

(b) Is there any other way to choose numbers to substitute x and y so that the product $(x + y)(x - y)$ will equal 0? If so, what is $x + y$?

(c) In general, if the product of two numbers is zero, what must be true about one of them?

(d) Consider Identity 2 for the Square of a Sum: $(x+y)^2 = x^2 + 2xy + y^2$. Is there a way to choose numbers to substitute for x and y so that the product xy equals 0?

(e) Is it ever possible that $(x + y)^2$ could equal $x^2 + y^2$? Explain your answer.

(f) Could $(x - y)^2$ ever equal $x^2 - y^2$? Explain your answer.

Just Jogging Learning Task

For distances of 12 miles or less, a certain jogger can maintain an average speed of 6 miles per hour while running on level ground.

1. If this jogger runs around a level track at an average speed of 6 mph, how long in hours will the jogger take to run each of the following distances? [Express your answers as fractions of an hour in simplest form.] (a) 3 miles (b) 9 miles (c) 1 mile (d) $\frac{1}{2}$ mile (e) $\frac{1}{10}$ mile

2. Analyze your work in Question 1. Each answer can be found by using the number of miles, a single operation, and the number 6. What operation should be used? Write an algebraic expression for the time it takes in hours for this jogger to run x miles on level ground at an average speed of 6 miles per hour.

3. Each day this jogger warms up with stretching exercises for 15 minutes, jogs for a while, and then cools down for 15 minutes. How long would this exercise routine take, in hours, if the jogger ran for 5 miles? [Express your answer as a fraction in simplest form.]

4. Let T represent the total time in hours it takes for this workout routine when the jogger runs for x miles. Write a formula for calculating T given x, where, as in Question 2, x is number of miles the jogger runs. Express the formula for T as a single algebraic fraction.

5. If the jogger skipped the warm-up and cool-down period and used this additional time to jog, how many more miles would be covered? Does this answer have any connection to the answer to question 4 above?

Suppose this same jogger decides to go to a local park and use one of the paths there for the workout routine one day each week. This path is a gently sloping one that winds its way to the top of a hill.

6. If the jogger can run at an average speed of 5.5 miles per hour up the slope and 6.5 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.

7. If the jogger can run at an average speed of 5.3 miles per hour up the slope and 6.7 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.

8. Write an algebraic expression for the total time, in hours, that it takes the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill if the jogger runs uphill at an average speed that is *c* miles per hour slower than the level-ground speed of 6 miles per hour and runs downhill at an average speed that is *c* miles per hour faster than the level-ground speed of 6 miles per hour. Simplify your answer to a single algebraic fraction. Verify that your expression gives the correct answers for Questions 6 and 7.

9. The average speed in miles per hour is defined to be the distance in miles divided by the time in hours spent covering the distance.

   (a) What is the jogger's average speed for a two mile trip on level ground?

   (b) What is the jogger's average speed for the two mile trip in question 6?

   (c) What is the jogger's average speed for the two mile trip in question 7?

   (d) Write an expression for the jogger's average speed over the two-mile trip (one mile up and one mile down) when the average speed uphill is *c* miles per hour slower than the level-ground speed of 6 miles per hour and the average speed downhill at an average speed that is *c* miles per hour faster than the level-ground speed of 6 miles per hour. Express your answer as a simplified algebraic fraction.

   (e) Use the expression in part (d) to recalculate your answers for parts (b) and (c)? What value of *c* should you use in each part?

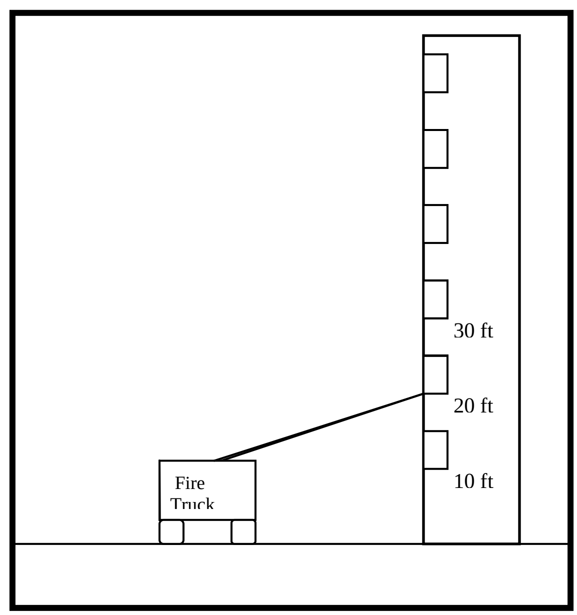
10. For what value of *c* would the jogger's average speed for the two-mile trip (one mile up and one mile down) be 4.5 miles per hour? For this value of *c*, what would be the jogger's average rate uphill and downhill?

Ladders Learning Task

Firefighters are important members of the community who not only fight fires but also participate in a variety of rescue activities. A ladder truck is an important tool that firefighters use to get water to and rescue people from heights above the ground floor. This activity explores relationships between the building floor firefighters need to reach and the length to which they need to extend the ladder mounted on the truck.

One metropolitan fire department has a truck with a 100 ft extension ladder. The ladder is mounted on the top of the truck. When the ladder is in use, the base of the ladder is 10 feet above the ground.

Suppose that a ladder truck is parked so that the base of the ladder is 20 feet from the side of an apartment building. Because there are laundry and storage rooms in the basement of the building, the base of the windows in first floor apartments are 10 feet above the ground, and there is a distance of 10 feet between the bases of the windows on adjacent floors. See the diagram below.



1. Find the length to which the ladder needs to be extended to reach the base of a window on the second floor. What is the exact answer? Would an approximation be more meaningful in this situation? Make an approximation to the nearest tenth of a foot and check your answer.

2. Find the length to which the ladder needs to be extended to reach the base of a window on the third floor.

3. Find the length to which the ladder needs to be extended to reach the base of a window on the fourth floor.

4. Let *n* represent the number of the floor that the ladder needs to reach. Write a function where the input is *n* and the output is the height *h*, in feet above the ground, of the base of windows on that floor. Does this function express a linear relationship?

5. Write a function where the input is *n* and the output is the length *L* of the ladder when it is extended to reach the base of a window on floor *n*. If possible, simplify this expression. Does this function express a linear relationship?

6. In this situation, what is the highest floor that the ladder could reach?

Planning for the Prom Culminating Task

April, Eric, Jason, Nicole, Ryan, and Trina are junior class officers at James K. Polk High School. As junior class officers, they also serve as the Executive Committee for planning the prom and give final approval for all arrangements for the event. The Prom Committee has arranged for the prom to be held at a country club in an adjacent county.

1. One issue to be decided about prom arrangements is the size of the dance floor. The prom will be in the carpeted country club ballroom. For dances, the club places a parquet dance floor over part of the carpet. The dance floor is laid in interlocking sections so there are several sizes that can be chosen. The country club requires a final decision on the size of the dance floor a month before the date of the prom so the committee wants to be prepared to base their decision on the latest possible information about the number of ticket sales.

   (a) The minimum size dance floor that the Prom Committee is considering is 30' by 30'. They assume that this size will allow 100 couples to be on the dance floor at once. How many square feet of dance space per couple does this size allow?

   To allow for more dancers, the country club can increase the dimensions of the dance floor in 5 foot increments in either direction; however, the committee would like to maintain a square shape.

   (b) Let n represent the number of 5' increases to the length of each side of the dance floor. Write an expression in terms of n for the additional square feet of dance floor, and then use algebraic identities to find a different but equivalent expression for the additional area. One expression should be a difference of areas, and the other should be a sum of areas.

   (c) Write an expression that the committee could use to estimate the total number of couples that can be on the dance floor when the width and length of the dance floor have been increased using n additions of 5' each. Assume that each couple needs the same square feet of dance space as in part (a) and that the committee will round down to the nearest whole number of couples if the expression gives a fractional number of couples.

(d) Suppose the committee choose to add enough 5' increments to increase the 30' side length of the dance floor by 50%. Find the total number of couples that can be on the dance floor with this increase in floor space. Calculate directly, and then verify that the expression in part (c) gives the same answer.

(e) Suppose the committee wants to consider allowing 5 square feet of dance floor space per person. How many couples would fit on the 30' by 30' dance floor? Redo parts (c) and (d) using 5 square feet of dance floor space per person.

(f) How does the number of couples change when allowing 9 square feet of dance floor per couple and increasing the length of the 30' side by 50%?

How does the number of couples change when allowing 10 square feet of dance floor per couple and increasing the length of the 30' side by 50%?

(g) Draw a geometric figure to explain why increasing the length of each side by 50% will more than double the number of couples that can be on the dance floor. Does your explanation depend on the sides starting at 30 feet each? Explain why or why not.

2. The Prom Committee plans to have all the seniors promenade before the Prom Queen and King are announced. They plan to lay out a promenade aisle along one of the diagonals of the dance floor by rolling out a "red carpet" of vinyl sheeting just before the promenade. Therefore, they also need to plan ahead for the length of the sheeting they will need.

(a) Write a radical for exact length of the diagonal of the 30' by 30' dance floor. Simplify the radical.

(b) Write a radical for exact length of the diagonal of the 40' by 40' dance floor. Simplify the radical.

(c) How does the length of the diagonal of a square compare to the length of a side of a square? Is this relationship evident in your answers to (a) and (b) above?

(d) Write an expression for the length of the diagonal across the dance floor, in feet, as a function of the number *n* of 5' extensions to the 30' by 30' square. Simplify the expression, if possible.

3. April, Eric, Jason, Nicole, Ryan, and Trina investigated getting a stretch limo to drive them and their dates on the evening of the prom. They found that a stretch limo for twelve people would be $150 per hour. They planned to meet at Trina's, go out to dinner, then to the prom, and return to Trina's for a party after the prom. As members of the prom Executive Committee they planned to stay for the full length of the prom so they concluded that they should reserve the limo for at least six hours.

(a) How much would it cost per couple if they rented a limo at $150 per hour for six hours?

Eric suggested that it might cost less and be more fun if they invited other members of the prom committee and their dates to join in and get a party bus together. He found that they could get a party bus that holds a maximum of 24 passengers at a cost $200 per hour as long as they rented the bus for at least six hours?

(b) How much more would it cost per couple if they rented the party bus for six hours but did not get any other couples to share the cost?

(c) Let *n* denote a number of couples, in addition to the members of the Executive Committee and their dates that could go on the party bus. What is the domain for *n*?

(d) Write an algebraic expression that can be used to define the cost, in dollars per couple, as a function of n?

(e) Letting n represent the number of additional couples that ride on the party bus, write a single rational expression for the difference in cost per couple to members of the Executive Committee for renting the party bus and the cost per couple to them for renting the stretch limo.

(f) Use the expression from (e) to write the cost difference as a function of n and investigate this function. Determine how many additional couples need to ride on the party bus so that, for the Executive Committee, the cost per couple is the same as renting the stretch limo.

(g) If April, Eric, Jason, Nicole, Ryan and Trina are sure that they can get three more couples to go with them and their dates and they want to keep their costs down, what would you advise them to do and why?

4. Nicole is wondering if six hours will be enough time to do all that they plan to do between the time the group leaves Trina's house and returns. She is especially concerned if they have estimated enough travel time for the trip from Trina's house to the restaurant, from the restaurant to the country club where the prom is being held, and from the prom back to Trina's house. In deciding on six hours for their transportation, they had roughly estimated an hour total for the travel.

   Nicole decided to be more precise in making an estimate. She checked the distances and decided to make some reasonable guesses about how fast they would be able to go. Since the restaurant and country club are each very near interstate exits, she included their distances from the interstate with the interstate mileage.

She assumed that they could average 40 mph for the 4 miles from Trina's house to the nearest interstate and 60 mph for the additional 16 miles on interstate highways to get to the restaurant. Their route from the restaurant to the prom will take them back along part of their route to the restaurant and then near an arena where there is a major concert scheduled for the evening of the prom. She took the heavy traffic of concert-goers into consideration and assumed they could average 40 mph for this 8 miles of Interstate. The concert will be over long before the prom, so she assumed that they will be able to go 60 mph along the 20 miles of interstate from the prom back to the Trina's exit and, late at night, will able to average 45 mph for the last 4 miles back to Trina's house.

(a) If all of their estimates of travel time are accurate, how long will they spend traveling from one location to another on prom night? How good is the rough estimate of travel time?

Nicole thinks that the biggest unknown in her assumptions is the effect of the concert traffic around the arena. She is concerned that they may not be able to average 40 mph for this 8 miles of the trip. She also wonders if she should assume that they will be able to cover the 4 miles from the interstate to Trina's house faster late at night.

(b) Let x represent a number of miles per hour slower that they might travel on the trip from the restaurant to the prom, Write a linear expression for the average speed in miles per hour on this part of the trip. Then use this expression to find a single rational expression for the time in hours that the whole trip will take if this is the speed from the restaurant to the prom and if the last segment of the trip home, from the interstate to Trina's, is covered at the same 40 miles per hour as earlier in the evening.

For the remaining parts assume that can go only 40 mph from the interstate back to Trina's house when they are coming home so that the above expression applies.

(c) Find the time in hours for the whole evening's travel if they drive from the restaurant to the prom at a rate that is 5 miles per hour slower than Nicole originally estimated. Repeat for 10 miles per hour slower.

(d) How could you use your rational expression from part (b) to find the total travel time for the evening if they can drive from the restaurant to the prom at a rate that is 5 miles per hour faster than Nicole originally estimated?

Teachers: Your feedback regarding the tasks in this unit is welcomed. After your students have completed the tasks, please send suggestions to Janet Davis (jdavis@doe.k12.ga.us).