Georgia Performance Standards Framework

Mathematics 1 - Unit 4
The Chance of Winning

Introduction: In this unit, students will calculate probabilities based on angles and area models, compute simple permutations and combinations, calculate and display summary statistics, and calculate expected values. They should also be able to use simulations and statistics as tools to answering difficult theoretical probability questions.

Enduring Understandings: By using the mathematical skills acquired from statistics and probability, students can better determine whether games of chance are really fair. They should also be able to use mathematics to improve their strategies in games.

Key Standards Addressed:

MM1D1 Students will determine the number of outcomes related to a given event.
- a. Apply the addition and multiplication principles of counting
- b. Calculate and use simple permutations and combinations

MM1D2. Students will use the basic laws of probabilities
- a. Find the probabilities of mutually exclusive events
- b. Find probabilities of dependent events
- c. Calculate conditional probabilities
- d. Use expected value to predict outcomes

MM1D3. Students will relate samples to a population
- a. Compare summary statistics (mean, median, quartiles, and interquartile range) from one sample data distribution to another sample data distribution in describing center and variability of the data distributions.
- b. Compare the averages of summary statistics from a large number of samples to the corresponding population parameters
- c. Understand that a random sample is used to improve the chance of selecting a representative sample.

MM1D4. Students will explore variability of data by determining the mean absolute deviation (the averages of the absolute values of the deviations).

RELATED STANDARDS ADDRESSED:

MM1G2. Students will understand and use the language of mathematical argument and justification.
- a. Use conjecture, inductive reasoning, deductive reasoning, counterexamples, and indirect proof as appropriate

Student Edition

Georgia Performance Standards Framework for Mathematics 1

MM1P1. Students will solve problems (using appropriate technology).

a. Build new mathematical knowledge through problem solving.
b. Solve problems that arise in mathematics and in other contexts.
c. Apply and adapt a variety of appropriate strategies to solve problems.
d. Monitor and reflect on the process of mathematical problem solving.

MM1P2. Students will reason and evaluate mathematical arguments.

a. Recognize reasoning and proof as fundamental aspects of mathematics.
b. Make and investigate mathematical conjectures.
c. Develop and evaluate mathematical arguments and proofs.
d. Select and use various types of reasoning and methods of proof.

MM1P3. Students will communicate mathematically.

a. Organize and consolidate their mathematical thinking through communication.
b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
c. Analyze and evaluate the mathematical thinking and strategies of others.
d. Use the language of mathematics to express mathematical ideas precisely.

MM1P4. Students will make connections among mathematical ideas and to other disciplines.

a. Recognize and use connections among mathematical ideas.
b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

MM1P5. Students will represent mathematics in multiple ways.

a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.

UNIT OVERVIEW

Students should already have knowledge that probabilities range from 0 to 1 inclusive. They should also be able to determine the probability of an event given a sample space. They should be able to calculate the areas of geometrical figures and measure an angle with a protractor.

Sometimes when studying probability, it is easier to understand how to find an answer by examining a smaller sample space. The wheel used on *Wheel of Fortune* has many different sections. It also has "lose a turn" and "bankrupt" which turns a simple probability problem into one that is much more complex. In addition, each section of the wheel may not have the same area; therefore, this type of spinner may be different from the ones that are familiar to students.

Permutations versus Combinations:
Students tend to confuse permutations with combinations. When teaching this portion of the unit, I would suggest integrating permutations with combinations with simple problems involving the multiplication principle. Students need many opportunities to decide which formula to use in which context prior to a unit assessment. You also may want to refrain from giving them the formula for permutations and combinations immediately. Instead, students should discover the patterns first before they see the formula. This should make the formulas more meaningful and help with retention.

When the sample space is too large to be represented by a tree diagram:
It's easy to write the sample space for flipping a coin 4 times and determining the probability that you have at least 2 heads. However, problems arise when you ask them to find the probability of at least 2 heads when flipping the coin 20 times since the sample size is very large $2^{20} = 1048576$. (Try listing those outcomes in a 50 minute class period!)

To solve this problem, you may have students explore patterns in smaller sample spaces. Have students draw the tree diagrams for 2 flips, 4 flips, 6 flips, etc. Ask students to examine the patterns in the sizes of the sample spaces to help them determine the size of the sample space for 20 flips. Have them find a strategy, based on these smaller sample sizes, to come up with a way to count "at least 2 heads for 20 flips."

When events are not equally likely:
In middle school, students may have only used tree diagrams for equally likely events (flipping a fair coin, rolling a fair die, etc.). If the events are equally likely, the branches of the tree diagram do not have to be labeled with the associated probabilities for students to get the correct probability. For example, suppose a fair coin is tossed twice. If a tree diagram is used to determine the probability of getting a "head" on the first flip and a "tail" on the second flip, students can easily see the sample space, $\{(H,H),(T,T),(H,T),(T,H)\}$, and realize that HT occurs once out of 4 times. Students can use the multiplication principle to confirm that the probability of HT is (.5)(.5)=.25. Thus, it would not matter whether the students labeled the branches of the tree diagram with the associated probabilities if the coin is fair.

Suppose that the coin is not fair. Suppose the probability of heads is .6 and the probability of tails is .4. Then the probability of HT = (.6)(.4) = .24 not .25. The sample space is still $\{(H,H),(T,T),(H,T),(T,H)\}$, but the probability of HT is no longer ¼ since the probability of heads is not the same as the probability of tails. To possibly avoid this problem, ask the students to label the branches tree diagram with their associated probabilities.

<u>When students cannot calculate the probability of an event:</u>
If students don't understand the theory, use simulations! Many adults as well as students struggle with probability. A good example of this is the classic "Monty's Dilemma" problem addressed in the "Ask Marilyn" column. The question was based on the popular "Let's Make a Deal" show. At one point on the show, there were 3 curtains. Behind one curtain was a great prize. Behind the other two curtains were awful prizes. The show's host, Monty, asked the player to pick a curtain. <u>After the player picked a curtain</u>, the host revealed a prize behind one of the other two curtains (not the good prize). The host then asked the player if he would like to stay or switch.

Marilyn stated that the player should switch because the probability that he picked the grand prize from the beginning was 1/3. So, the probability of winning would be 2/3 if the player switched.

She received many letters, some from mathematicians, claiming that it would not matter if the player stayed or switched…the probability of winning now changed to 1/2 since one bad prize was revealed.

She then asked middle schools across the country to simulate this and send her the results. The results stated that it was better to switch.

Many times, "real life" probability is difficult to compute or hard to understand. That is why it's so important to be able to perform simulations. With computers and graphing calculators readily available, simulations are easy to perform and are not time consuming.

<u>Understanding conditional probability:</u>
Although there are other methods, I typically teach my students the following two techniques to solve conditional probability problems.

Technique #1: Think of the sample space described. For example, suppose the question reads, "Given a person rolls an even number on a die, what is the probability that the die lands on a 2?" I ask students to list the sample space described. It is not all possible outcomes on the die because we know that the person rolled an even number. The sample space is just $\{2,4,6\}$. Therefore, the probability is 1/3.

Technique #2: Use a tree diagram and the conditional probability formula.

The formula for conditional probability is

P(A given B) = P(A and B)/P(B). It's symbolically written as such

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ where $\cap$ stands for the intersection of sets A and B.

This formula for the previous problem would be much more difficult than technique #1. Using the formula, the numerator would be

$P(A \cap B) = P(\text{rolls a 2 and rolls an even number}) = \dfrac{1}{6}$.

The denominator would be $P(B) = P(\text{even number}) = \dfrac{3}{6}$.

We would get the same result, $\dfrac{\frac{1}{6}}{\frac{3}{6}} = \dfrac{1}{3}$ but it would be a little more difficult and time consuming for students.

The formula, however, does have its merit. Sometimes, it is not easy or possible to list the sample space. In that case, the formula is necessary.

A problem which requires the formula might read, "Suppose a student knows 30% of the class material without studying for an upcoming multiple choice test which has 15 questions (4 possible answers per question). Suppose the student does not study for the test. If she provides the correct answer on the test, what is the probability that she strictly guessed?"

I have found tree diagrams very helpful to solve these types of conditional probability problems. Students should already be familiar with constructing tree diagrams from middle school.

Formulas and Definitions

- Addition Rule for mutually exclusive (disjoint) events:
  $P(A \text{ or } B) = P(A) + P(B)$

- Addition Rule for sets that are not mutually exclusive:
  $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$

- Census: A census occurs when everyone in the population is contacted.

- Conditional Probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

- Combinations: ${}_nC_r = \dfrac{n!}{r!(n-r)!}$

- Complement: This refers to the probability of the event not occurring $P(A^c) = 1 - P(A)$

- Dependent: Two events are dependent when the outcome of the first event affects the probability of the second event. For example, suppose two cards are drawn from a standard deck of 52 cards without replacement. If you want the probability that both cards are kings, then it would be $\dfrac{4}{52} * \dfrac{3}{51}$. If a king was drawn first, then there would only be 3 kings left out of 51 cards since the first king was not put back in the deck. Hence, the probability of drawing a king on the second draw is different than the probability of drawing a king on the first draw, and the events are dependent.

- Expected Value: The mean of a random variable X is called the expected value of X. It can be found with the formula $\sum_{i=1}^{n} X_i P_i$ where P_i is the probability of the value of X_i.
  For example: if you and three friends each contribute \$3 for a total of \$12 to be spent by the one whose name is randomly drawn, then one of the four gets the \$12 and three of the four gets \$0. Since everyone contributed \$3, one gains \$9 and the other three looses \$3. Then the expected value for each member of the group is found by (.25)(9) +(.75)(-3) = 0. That is to say that each pays in the \$3 expecting to get \$0 in return. A game or situation in which the expected value of the profit for the player is zero (no net gain nor loss) is commonly called a "fair game." However, if you are allowed to put your name into the drawing twice, the expected value is (.20)(9)+(.80)(-3) = -.60. That is to say that each pays in the \$3 expecting to get -\$.60 in return.

- Fair: In lay terms it is thought of as "getting the outcome one would expect," not as all outcomes are equally likely. If a coin (which has two outcomes...heads or tails) is fair, then the probability of heads = probability of tails = 1/2. If a spinner is divided into two sections...one with a central angle of 120 degrees and the other with an angle of 240 degrees, then it would be fair if the probability of landing on the first section is 120/360 or 1/3 and the probability of landing on the second section is 240/360 or 2/3.

- Independent: Two events are independent if the outcome of the first event does not affect the probability of the second event. For example, outcomes from rolling a fair die can be considered independent. The probability that you roll a 2 the first time is $\frac{1}{6}$. If you roll the die again, there are still 6 outcomes, so the probability that you roll a 2 the second time is still $\frac{1}{6}$.

- Measures of Center
  - Mean: The average = $\frac{\sum_{i=1}^{n} X_i}{N}$. The symbol for the sample mean is $\bar{X}$. The symbol for the population mean is μ_X.
  - Median: When the data points are organized from least to greatest, the median is the middle number. If there is an even number of data points, the median is the average of the two middle numbers.
  - Mode: The most frequent value in the data set.

- Measures of Spread (or variability)
  - Interquartile Range: $Q_3 - Q_1$ where Q_3 is the 75th percentile (or the median of the second half of the data set) and Q_1 is the 25th percentile (or the median of the first half of the data set).
  - Mean Deviation: $\frac{\sum |X_i - \bar{X}|}{N}$ where X_i is each individual data point, $\bar{X}$ is the sample mean, and N is the sample size

- Multiplication Rule for Independent events: $P(A\ and\ B) = P(A)P(B)$

- Mutually Exclusive: Two events are mutually exclusive (or disjoint) if they have no outcomes in common.

- Parameters: These are numerical values that describe the population. The population mean is symbolically represented by the parameter μ_X. The population standard deviation is symbolically represented by the parameter σ_X.

- Permutations: ${}_nP_r = \frac{n!}{(n-r)!}$

- Random: Events are random when individual outcomes are uncertain. However, there is a regular distribution of outcomes in a large number of repetitions. For example, if you flip a fair coin 1000 times, you will probably get tails **about** 500 times. But, you probably won't get HTHTHTHT or even HTHTHHTTHHTT when you flip the coin, so the outcome is uncertain for **each** flip. Or, if you roll two dice and record the sums 1000 times, you will probably get **about** 167 sums of 7, 139 sums of 6, etc. which are the expected values (expected value for a sum of 7 is 6/36*1000 =167). Hence, we will have a regular distribution of outcomes. However, since rolling two fair dice is a random event, we won't know what sum our dice will give on **each** roll.

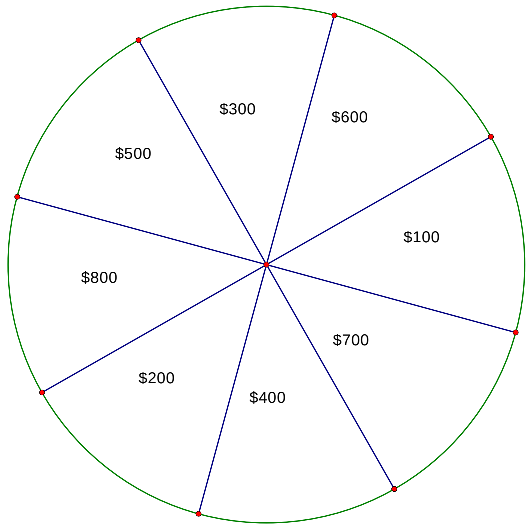
- Sample: A subset, or portion, of the population.

- Sample Space: The set of all possible outcomes.

- Statistics: These are numerical values that describe the sample. The sample mean is symbolically represented by the statistic $\overline{X}$. The sample standard deviation is symbolically represented by the statistic s_x.

TASKS: The remaining content of this framework consists of student tasks. The first is intended to launch the unit by developing a basic understanding of what random and fair are. Tasks are designed to allow students to build their own understanding through exploration follow. The tasks fall under three basic considerations: Wheel of Fortune, True or False and Testing, and Yahtzee. The students will find these tasks engaging, mathematically rich and rigorous.

Spinner 1 Learning Task

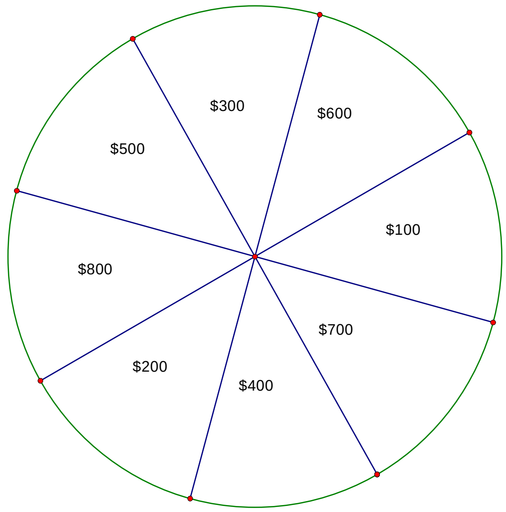


Make an arrow for the spinner above so that the spinner is useable. You can use brads and a paper clip for an arrow. Is this homemade spinner fair? How can you tell?

Calculate the following probabilities for the above spinner (assuming the spinner is fair).

1) What is the probability of obtaining $800 on the first spin?
2) What is the probability of obtaining $400 on the first spin
3) Is it just as likely to land on $100 as it is on $800?
4) What is the probability of obtaining at least $500 on the first spin?
5) What is the probability of obtaining less than $200 on the first spin?
6) What is the probability of obtaining at most $500 on the first spin?
7) If you spin the spinner twice, what is the probability that you will have a sum of $200?
8) If you spin the spinner twice, what is the probability that you will have a sum of at most $400?
9) If you spin the spinner twice, what is the probability that you will have a sum of at least $1500?
10) If you spin the spinner twice, what is the probability that you will have a sum of at least 300?
11) Given that you land on $100 on the first spin, what is the probability that the sum of your two spins will be $200?
12) Given that you land on $800 on the first spin, what is the probability that the sum of your two spins will be at least $1000?

Spinner Learning Task 2



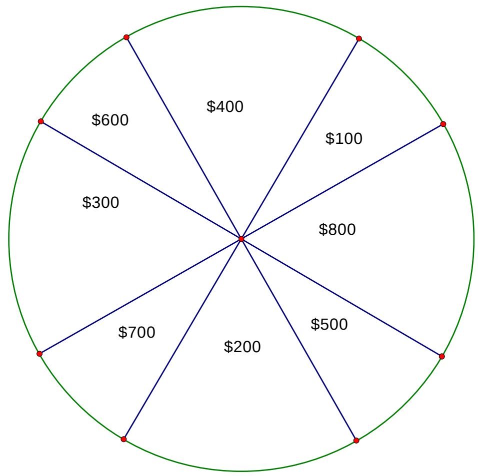
1) With the above spinner, assuming it is fair, how much money would you expect to receive each time you spin the spinner?

2) If you spin the spinner twice, how much money, on average, would you expect to receive?

3) If you spin the spinner 10 times, how much money, on average, would you expect to receive?

4) On average what is the fewest number of spins it will take to accumulate $1000 or more?

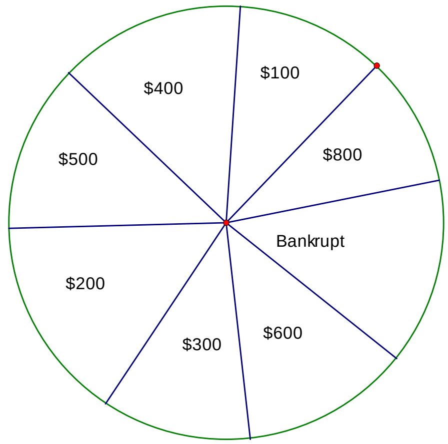
Spinner Learning Task 3



Based on this spinner, calculate the following:

1) What is the probability of obtaining $800 on the first spin?
2) What is the probability of obtaining $500 on the first spin?
3) Is it just as likely to land on $100 as it is on $800?
4) What is the probability of obtaining at least $500 on the first spin?
5) What is the probability of obtaining less than $200 on the first spin?
6) What is the probability of obtaining at most $500 on the first spin?
7) If you spin the spinner twice, what is the probability that you will have a sum of $200?
8) If you spin the spinner twice, what is the probability that you will have a sum of at most $400?
9) If you spin the spinner twice, what is the probability that you will have a sum of at least $1500?
10) Given that you landed on $100 on the first spin, what is the probability that the sum of your two spins will be $200?
11) Given that you landed on $800 on the first spin, what is the probability that the sum of your two spins will be at least $1500?
12) If you spin the spinner once, how much money, on average, would you expect to receive? Compare this answer to the simulation in Task #2. (MM1D3(b))
13) If you spin the spinner twice, how much money, on average, would you expect to receive?
14) If you spin the spinner 10 times, how much money, on average, would you expect to receive?

Spinner Learning Task 4



1) What is the probability of obtaining $800 on the first spin?
2) What is the probability of obtaining $500 on the first spin?
3) Is it just as likely to land on $100 as it is on $800?
4) What is the probability of obtaining at least $500 on the first spin?
5) What is the probability of obtaining less than $200 on the first spin?
6) What is the probability of obtaining at most $500 on the first spin?
7) If you spin the spinner twice, what is the probability that you will have a sum of $200?
8) If you spin the spinner twice, what is the probability that you will have a sum of at most $200?
9) If you spin the spinner twice, what is the probability that you will have a sum of at least $1500?
10) Given that you landed on $100 on the first spin, what is the probability that the sum of your two spins will be $200?
11) Given that you landed on $800 on the first spin, what is the probability that the sum of your two spins will be at least $1500?
12) If you spin the spinner once, how much money, on average, would you expect to receive?

You may want to do some simulations to answer problems 13 and 14. If you land on bankrupt, then you lose all of the money that you have accumulated.

13) If you spin the spinner twice, how much money, on average, would you expect to receive?

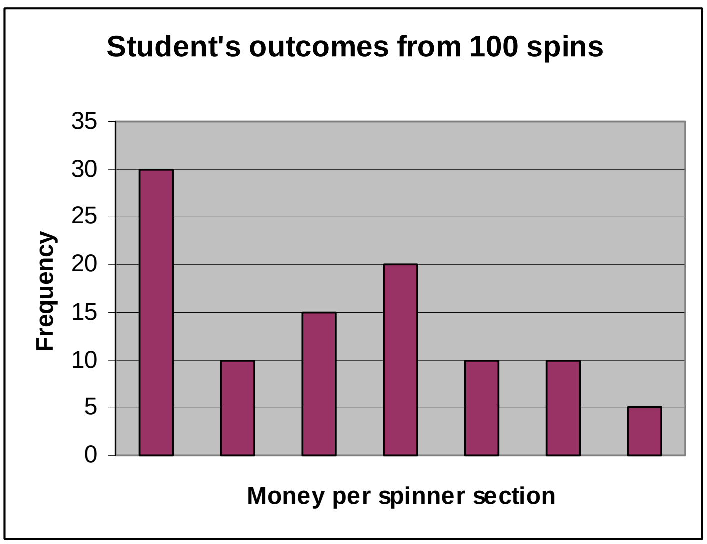
14) If you spin the spinner three times, how much money, on average, would you expect to receive?

15) How many times on average would you have to spin until you land on "bankrupt?

Spinner Learning Task 5

A student created a spinner and recorded the outcomes of 100 spins in the table and bar graph below.

Amount of money on each section of the spinner	$0	$100	$200	$300	$400	$500	$600
Number of times that the student landed on that section	30	10	15	20	10	10	5



Based on the table and graph above, calculate the experimental probabilities of landing on each section of the spinner. Assuming that these experimental probabilities are the theoretical probabilities, use these probabilities to draw what the spinner would look like below.

For the rest of the problems, assume that the experimental probabilities are the theoretical probabilities:

Calculate the average amount of money a person would expect to receive on each spin of the spinner.

Calculate the probability that you will receive at least $400 on your first spin.

Given you land on $300 the first time, what is the probability that the sum of your first two spins is at least $600?

What is the probability that the sum of two spins is $400 or less?
You propose a game:

A person pays $350 to play. If the person lands on $0, $200, $400, or $600, then they get that amount of money and the game is over. If the person lands on $100, $300, $500, then they get to spin again, and they will receive the amount of money for the sum of the two spins.

In the long run, would the player expect to win or lose money at this game? If the player plays this game 100 times, how much would he/she expect to win or lose?

Spinner Task 6
Design and make a workable spinner. Use your spinner to compare experimental vs. theoretical probabilities. Use your spinner to calculate expected values, probabilities of mutually exclusive events and conditional probabilities. Based on your comparisons of the experimental and theoretical probabilities, answer the question, “is your spinner fair?” The following rubric will be used to grade your spinner and paper. You must turn in the spinner along with a paper that describes the experimental and theoretical probabilities above.

Maximum Points awarded if you:	Moderate Points awarded if you:	Minimum Number of Points Awarded if you:	Points Awarded
A durable and workable spinner is made with at least 6 sections of unequal areas.	A durable and workable spinner is made with less than 6 sections of unequal area	A durable and workable spinner is made with less than 4 sections of unequal area.	
The student correctly calculates the theoretical probability of landing on each section of the spinner. The process of calculating the probability is clearly communicated in the paper.	The student correctly calculates the theoretical probability of landing on each section of the spinner. The probabilities are given in the paper without a clear explanation of how they are found.	The student incorrectly calculates the theoretical probability of landing on each section of the spinner.	
The student displays a dot plot of 100 spins. The experimental probability is calculated for landing on each section. The experimental probability is explicitly compared to the theoretical probability in the paper and is used to determine if the spinner is working properly.	The student displays a dot plot of 100 spins. The experimental probability is calculated for landing on each section. The paper lacked a good comparison between the theoretical and experimental probabilities.	The student did not display a dot plot of 100 spins, or the paper did not compare the theoretical and experimental probabilities.	
The student correctly calculates the expected value, the average amount a person would receive in one spin, using the theoretical probabilities. All work is shown and the process is clearly communicated in the paper.	The student correctly calculates the expected value, the average amount a person would receive in one spin, using theoretical probabilities. The process is not communicated clearly in the paper.	The student incorrectly calculates the expected value.	
The student accurately defines mutually exclusive events. The student makes up and correctly solves two probability questions based on his/her spinner that accurately reflect this concept. All work is shown in the paper of how the problems are solved.	The student accurately defines mutually exclusive events, but the two probability problems do not reflect this concept. The problems are solved correctly and all work is shown.	The student accurately defines mutually exclusive events but the two probability problems do not reflect this concept. The problems are solved correctly and no work is shown, or they are not solved correctly.	

The student accurately defines conditional probability. The student makes up and correctly solves two probability problems based on his/her spinner that accurately reflect this concept. All work is shown in the paper of how the problems are solved.	The student accurately defines conditional probability. The student makes up and correctly solves two probability problems that do not reflect this concept. All work is shown.	The student accurately defines conditional probability, but the two problems do not reflect this concept. The problems are solved correctly and no work is shown, or they are not solved correctly.	
The student performs a simulation to answer the question, "If I spin the spinner three times, how much money should I expect to accumulate?" At least 30 simulations are performed to answer this question. The student describes how he/she performs the simulations. A dot plot or histogram is displayed of the simulations. The mean and mean deviation are calculated, or the median and interquartile range are calculated. Both of the summary statistics are used to answer the question. Finally, the mean of the simulations are compared to the theoretical expected value for three spins.	The student does most but not all of what was described in the first column.	The student did some of what was described in the first column.	
Explanation of mathematics is written in complete sentences and in an organized manner.	Explanation of mathematics is not written in complete sentences but is in an organized manner.	Explanation of mathematics is not written in complete sentences nor is in an organized manner.	
The paper is written neatly or typed.	The paper is not written neatly, but it was readable.	It is very hard to read because it is not neat.	

Spinner Learning Task 7

Suppose you are playing "Wheel of Fortune" and you are the first player to spin the wheel. Is it likely that you will solve the puzzle (provided you never land on "bankrupt" or "lose a turn")? In order to answer this question, calculate the following probabilities. Use those probabilities in your explanation.

1. Suppose you are the first player to spin the wheel, and you do not know the phrase, what is the probability that you guess a correct letter?
2. If it turns out that no letters in the word are repeated, what is the probability that the 2^{nd} letter you guess is correct (if you still do not know the phrase)?
3. So, if you have a 5 letter word, and no letters in the word are repeated, what is the probability that you guess all 5 correct (provided that you still have no idea what the word is even after the 4^{th} letter)?
4. Suppose that you are the 3^{rd} player to spin the wheel. You know that "S" and "T" are not in the phrase since those were the guesses of the first two players. What is the probability that you guess all 5 letters correctly if you never know what the word is?

Spinner Learning Task 8

Suppose you are playing the bonus round on the game show, "Wheel of Fortune." If you were allowed to pick any 8 consonants and any 2 vowels, which letters would you pick?

In a small group, play the "bonus round" from "Wheel a Fortune." This is a modified version of hangman. Let one member of your group come up with a phrase. You are to use only the 8 consonants and 2 vowels that you pick. Record the time it takes you to guess the phrase (do not take more than 3 minutes per phrase). Perform this simulation within your group 5 times. Record the length of time it took you to guess each of the 5 phrases.

Record the class data below. Calculate the 5 number summary and draw a box plot. Are there outliers? What is the interquartile range?

Calculate the mean and mean deviation of the class data.

Which is a better measure of center to use, the mean or the median? Why?

Which is a better measure of spread to use, the interquartile range (IQR) or the mean deviation? Why?

Now, use statistics to determine which letters and consonants are used the most in the English language.

- Prior to beginning this simulation, make a tally sheet. Write down all 26 letters to the alphabet in a vertical column on your notebook paper.
- Next, open a book/novel, close your eyes, and put your finger somewhere on the page. Begin at that spot and count how many A's, B's, C's, etc. occur in the first 150 letters that they see. Tally on the notebook paper.

Based on your simulation, answer the following questions:

- Which 8 consonants and 2 vowels are used the most often in the English language?
- Compare your answer with your classmate. Did you pick the same letters?
- Write the class data below. Compute the percent of A's, percent of B's, percent of C's, etc. from the class data.
- Compare your individual answer to the class answer.

Answer the question, "which 8 consonants and 2 vowels are used most often in the English language?"

Using the 8 consonants and 2 vowels that are used most frequently in the English language, play the bonus round of wheel of fortune again within your group five times. Record the time it takes you to guess the phrase (do not take more than 3 minutes per phrase).

The teacher will collect the class data. Using the class data, calculate the 5 number summary (minimum, 1st quartile, median, 3rd quartile, and maximum) and draw the associated box plot.

Compare and contrast the two box plots (old box plot before we knew the most frequently used letters with this box plot). In you explanation, you should compare the centers (medians), the IQR's (interquartile range), and the shapes (skewed or symmetric). You should then use these values to answer the following questions:

1. Did you save time today by using the letters we found to be used most often? Explain.
2. It took more than ______ seconds to answer 25% of the puzzles when we randomly provided the letters. It took more than _____ seconds to answer 25% of the puzzles when we used the most frequently used letters.
3. We answered 25% of the puzzles in less than _______seconds when we randomly guessed the letters. We answered 25% of the puzzles in less than _________ seconds when we used the most frequently used letters.

4. It took more than ________ seconds to answer half of the problems when we randomly provided the letters. It took more than ______ seconds to answer half of the problems when we used the most frequently used letters.
5. The bonus round only allows the player about 10 seconds to guess the phrase. Based on that, would we win more often or less often by randomly guessing or by using the frequently chosen letters?

<u>Spinner Learning Task 9</u>

Susan played the bonus round of wheel of fortune 30 times. She recorded how long it took her to guess the phrase to the nearest second. The following are the lengths of time it took her to guess each phrase correctly:

10, 11, 11, 12, 12, 12, 13, 13, 14, 14, 15, 15, 15, 15, 17, 18, 19, 21, 24, 24, 24, 26, 28, 31, 33, 34, 35, 35, 37, 40

Monique also played the bonus round of wheel of fortune 25 times. She recorded how long it took her to guess the phrase to the nearest second. The following are the lengths of time it took her to guess each phrase correctly:

12, 13, 13, 13, 13, 14, 14, 14, 14, 14, 15, 15, 15, 15, 15, 15, 16, 16, 16, 17, 17, 17, 18, 18, 55

a) Graph the two distributions below. Which measure of center (mean or median) is more appropriate to use and why? Calculate that measure of center.

b) Comment on any similarities and any differences in Susan’s and Monique’s times. Make sure that you comment on the variability of the two distributions.

c) If you are only allowed 15 seconds or less to guess the phrase correctly in order to win, which girl was more likely to win and why?

d) If Susan found that she could have guessed each phrase 3 seconds faster if she had chosen a different set of letters, would that have made any difference in your answer to part c? Why/why not?

Testing Learning Task 1

Today you are going to determine how well you would do on a true/false test if you guessed at every answer.

Take out a sheet of paper. Type randint(1,2) on your calculator. If you get a 1, write "true." If you get a 2, write "false."

Do this 20 times.

Before the teacher calls out the answers, how many do you expect to get correct? Why?

Grade your test. How many did you actually get correct? Did you do better or worse than you expected?

Make a dot plot of the class distribution of the total number correct.

Calculate the mean and median of your distribution. Which measure of center should be used based on the shape of your dot plot?

Calculate the mean deviation and the IQR. What do these numbers represent? Which measure of variability should be used based on the shape of your dot plot?

Based on the class distribution, what percentage of students passed?

Calculate the probabilities based on the dot plot:

1. What is the probability that a student got less than 5 correct?
2. What is the probability that a student got exactly 10 correct?
3. What is the probability that a student got between 9 and 11 correct (inclusive)?
4. What is the probability that a student got 10 or more correct?
5. What is the probability that a student got 15 or more correct?
6. What is the probability that a student passed the test?
7. Is it more likely to pass or fail a true/false test if you are randomly guessing?
8. Is it unusual to pass a test if you are randomly guessing?

Calculate the theoretical probabilities for problems #9-16.

9. What is the probability that a student got less than 5 correct?
10. What is the probability that a student got exactly 10 correct?
11. What is the probability that a student got between 9 and 11 correct (inclusive)?
12. What is the probability that a student got 10 or more correct?
13. What is the probability that a student got 15 or more correct?

How do the theoretical probabilities compare to the experimental probabilities?

<u>Testing Learning Task 2:</u>

Suppose there is a 5 question multiple choice test. Each question has 4 answers (A, B, C, or D). If you are strictly guessing, calculate the following probabilities:

1. P(0 correct) =
2. P(1 correct) =
3. P(2 correct) =
4. P(3 correct) =
5. P(4 correct) =
6. P(5 correct) =

Draw a histogram of the probability distribution for the # of correct answers. Label the x-axis as the number of correct answers. The y-axis should be the probability of x.

Based on the distribution, how many problems do you expect to get correct?

Based on the distribution, how likely is it that you would pass if you were strictly guessing? (Calculate the probability of getting 4 or 5 correct.)

What is the probability that you will get less than 3 correct?

What is the probability that you will get at least 3 correct?

Now let's look at tests, such as the SAT, when you are penalized for guessing incorrectly. Suppose you have a multiple choice test with five answers (A, B, C, D, or E) per problem. Then, the probability your guess is correct = 1/5. And the probability that your guess is incorrect is 4/5.

Suppose the test that you are taking will penalize you by ¼ of a point if you guess incorrectly. Test scores will be rounded.

If you strictly guess and get exactly 4 correct and 6 incorrect, what would be your score?

If you take a 10 question test and know that 8 questions are correct, should you guess the answers for the other two questions?

If you take a 10 question test and know that 6 questions are correct, should you guess the answers for the other 4 questions?

Given that you answered all 10 questions and you knew that 6 were correct, answer the following questions:

> If you can eliminate one of the answers for each of the 4 questions for which you are guessing, what would your percentage score be?
>
> If you can eliminate two of the answers for each of the 4 questions for which you are guessing, what would your percentage score be?
>
> If you can eliminate three of the answers for each of the 4 questions for which you are guessing, what would your percentage score be?

<u>Testing Learning Task 3</u>

Have you ever taken a multiple choice test when you may have had 4 or 5 "C's" in a row and thought that you made a mistake? Did the teacher intentionally put 4 C's in a row, or did you miscalculate? Or, could this randomly happen?

This leads to the question we will answer today, "Can a person/teacher really be random with the questions when he makes up a 50 question True/False test?"

Let's try. On a piece of paper, you make up the answer key to a 50 question true/false test. All you need to do is randomly write down T (for true) or F (for false) such as: TTFFTF….etc.

Now, on the bottom half of the paper, let your calculator generate your answer key. Type randint(0,1). "0" will stand for true, and "1" will stand for false. Press "enter" 50 times and record the outcomes of your calculator such as "00010110….".

Count the longest string of consecutive T's for both sets of data. For example, if a student had FFTTTTTTFFTTFFTF…., then her longest string of T's may have been 5.

Count the longest string of consecutive T's that you recorded when you made up the answer key. For example, if you had FFTTTTTTFFTTFFTF…., then your longest string of T's may have been 5. Ask each student the length of their longest string of T's. Make a dotplot of the distribution on the board. Find the center (mean or median) and the spread (mean deviation, IQR, or range).

Now, count the longest string of consecutive T's that you recorded when your calculator made up the answer key. Make a dot plot of the distribution on the board. Find the center (mean or median) and the spread (mean deviation, IQR, or range).

Compare the two distributions. Does one distribution usually have a longer string of "T's" than the other? On average, what is the longest string of "T's" that you would expect to see on a true/false test if the answers were truly placed in random order?

Now, do the same for a 50 question multiple choice test with 4 answers per problem (A, B, C, D). "How long would you expect the longest string of "C's" to be." Record guesses in a dot plot on the board.

Use your calculator to randomly generate the answer key to the 50 question test. Enter, radiant (1,4). Let "1" be "A", "2" be "B", "3" be "C", and "4" be "D". Each student should record their 50 answers and then count the longest string of "C's" that they have. Make a dot plot of this distribution on the board.

Calculate and compare the center and spread of the two distributions.

Is it likely that the teacher was random if he put 7 "C's" in a row on a test?
Is it likely that the teacher was random if he never put two consecutive letters in a row on a test?

Testing Learning Task 4

A teacher makes up a 5 question multiple choice test. Each question has 5 answers listed "a-e." A new student takes the test on his first day of class. He has no prior knowledge of the material being tested.

a) What is the probability that he makes a 100 just by guessing?

b) What is the probability that he only misses 2 questions?

c) What is the probability that he misses more than 2 questions?

d) Let X= the number correct on the test. Make a graphical display of the probability distribution below. Comment on its shape, center, and spread.

e) Based on your distribution, how many questions should the new student get correct just by randomly guessing?

f) The answers to the test turned out to be the following:
   1. A
   2. A
   3. A
   4. A
   5. C

   Do you think that the teacher randomly decided under which letter the answer should be placed when she made up the test? Explain.

Testing Learning Task 5

Earlier, we found the probability that the student passed a multiple choice test just by random guessing. However, we know that students usually have a little more knowledge than that, even when they do not study, and consequently do not guess for all problems.

Suppose that a student can retain about 30% of the information from class without doing any type of homework or studying. If the student is given a 15 question multiple choice test where each question has 4 answer choices (a, b, c, or d), then answer the following questions:

1. What is the probability that the student gives the correct answer on the test? What would be her percentage score on a 15 question test?
2. Given she provides the correct answer on the test, what is the probability that she strictly guessed?

Survey Learning Task

In small groups, work together to solve the following problem:

A student conducted a survey with a randomly selected group of students. She asked freshmen, sophomores, juniors, and seniors to tell her whether or not they liked the school cafeteria food. The results were as follows:

	Freshmen	Sophomores	Juniors	Seniors
Liked food	85	50	77	82
Did not like food	44	92	56	78

Using the table above, calculate the following probabilities:

1. What is the probability that the randomly selected student was a freshman?
2. What is the probability that the randomly selected student was either a junior or senior?
3. What is the probability that the randomly selected student was not a sophomore?
4. If you knew that the student interviewed was a freshman, what is the probability that the student liked the cafeteria food?
5. If you knew that the student interviewed was a junior or senior, what is the probability that the student did not like the cafeteria food?
6. If you knew that the student did not like the cafeteria food, what is the probability that the student was not a freshman?

Medical Learning Task

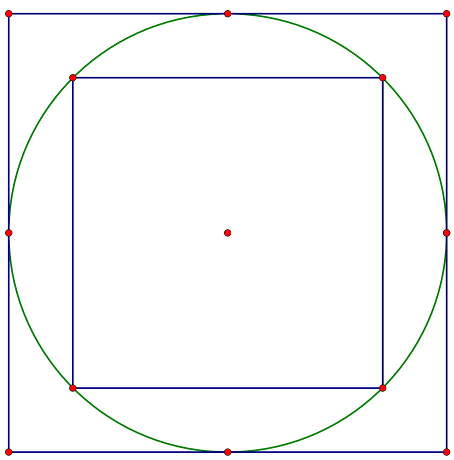
Work in small groups to solve the following:

> A patient is tested for cancer. This type of cancer occurs in 5% of the population. The patient has undergone testing that is 90% accurate and the results came back positive. What is the probability that the patient actually has cancer?
>
> (For help, the question asks P(cancer given the test is positive))

Area Learning Task

Let the students work together to try to solve the following problem:

A circle is inscribed within a square having each side of length 2 units. A smaller square is inscribed within the circle such that the corners of the square intersect the circle.



Length of side = 2

1. If you throw a dart at the board, and it lands in the large square, what is the probability that it lands in the circle?
2. If you throw a dart at the board, and it lands in the large square, what is the probability that it lands in the small square.
3. If you throw a dart at the board and it lands in the circle, what is the probability that it does not land in the small square?

Choose a point at random in the rectangle with boundaries $-1 \le x \le 1$ and $0 \le y \le 3$. This means that the probability that the point falls in any region within the square is the area of that region. Let X be the x-coordinate and Y be the y-coordinate of the randomly chosen point. Find the following:

a) P(Y>1 and X>0)
b) P(Y>2 or X>0)
c) P(Y>X)
d) P(Y>2 given Y>X)

Card Learning Task

Given a standard deck of 52 cards which consists of 4 queens, 3 cards are dealt, without replacement.

1. What is the probability that all three cards are queens?

2. Let the first card be the queen of hearts and the second card be the queen of diamonds. Are the two cards independent? Explain.

3. If the first card is a queen, what is the probability that the second card will not be a queen?

4. If the first two cards are queens, what is the probability that you will be dealt three queens?

5. If two of the three cards are queens, what is the probability that the other card is not a queen?

6. Answer questions #1 and #2 if each card is replaced in the deck (and the deck is well shuffled) after being dealt.

Marble Learning Task

So far, we have looked at mostly independent events. For example, each spin of the spinner can be considered as independent because the outcome of the 2nd spin does not rely on the outcome of the first spin. The same can be said about rolling dice. Each roll is independent of each other.

Today we will look at calculating dependent probabilities. When you sample without replacement, then the probabilities change. For example, suppose you have a deck of 52 cards. If I ask you what is the probability of drawing a queen, you would tell me 4/52. Now, suppose you drew a queen but did not replace the card. If I asked you "what is the probability of drawing a queen," you would now tell me "3/51." Note that this probability depends on the outcome of the first draw. Therefore, the two events are dependent.

#1: There are 21 marbles in a bag. Seven are blue, seven are red, and seven are green. If a blue marble is drawn from the bag and not replaced, what is the probability that:

a) A second marble drawn at random from the bag is blue?
b) A second marble drawn from the bag is blue or green?
c) A second marble drawn from the bag is not blue?

#2: There are 21 marbles in a bag. Seven are blue, seven are red, and seven are green. If the marbles are not replaced once they are drawn, what is the probability:

a) Of drawing a red marble and then a blue marble?
b) Of drawing a red marble, then a blue marble, then a green marble?
c) Of drawing a red marble or a blue marble and then a green marble?
d) Of drawing a red marble given that the first marble drawn was red?

#3: Use the table below to answer the questions:

A student conducted a survey with a randomly selected group of students. She asked freshmen, sophomores, juniors, and seniors to tell her whether or not they liked the school cafeteria food. The results were as follows:

	Freshmen	Sophomores	Juniors	Seniors
Liked food	85	50	77	82
Did not like food	44	92	56	78

a) What is the probability that a randomly selected student is a freshman?
b) What is the probability that a randomly selected student likes the food?
c) What is the probability that randomly selected student is a freshman and likes the food?
d) If the randomly selected student likes the food, what is the probability that he/she is a freshman?
e) Are the events "freshman" and "likes food" independent or dependent?

Dice Learning Task

1. Roll the two dice (one red and one green) 100 times. Record your outcomes on a piece of notebook paper as below:

   Red Die **Green Die** **Sum of Two Dice**

   Tally how many times that you rolled a sum of 2, 3, 4, …, 12.

2. Create a histogram based on the sums. The x-axis should be labeled "sum of two dice," and the y-axis should be labeled "frequency."
3. From your histogram, compute the mean and mean deviation.
4. Pool class data. Graph the histogram for the class. What is the shape of the histogram? How does your histogram for your data compare to the class histogram of the class data? Compute the mean and the mean deviation of the class histogram and compare it to your summary statistics.
5. Convert the y-axis of the class data from "frequency" to "probability." Ask the students if there is any difference in the shape, center, or spread after the conversion is made.
6. Based on the class histogram (experimental probability), compute the following probabilities:
   1) $P(sum = 5) =$ ________
   2) $P(sum \le 4) =$ ________
   3) $P(sum > 4) =$ ________
   4) $P(sum > 4 \text{ or } sum = 2) =$ _________
   5) What is the probability that the sum = 4 if the first die was a 3?_______
   6) Now compute the theoretical probabilities of the sum of two dice.
   7) Draw the theoretical probability distribution on your paper. The x-axis should be labeled "the sum of the two dice," and the y-axis should be labeled as the probability (instead of the frequency). Find the mean, mean deviation, and the answers to the probability questions in #5 for the theoretical distribution. Compare the experimental and theoretical distributions.

Simulation Learning Task

Instead of using real dice, today you will use the graphing calculator to simulate rolling 3 dice. On the TI-83 and TI-84, select the "Math" button. Next, select the "PRB" menu and choose "randInt". On the home screen, randInt(, should appear. Type in the following: randInt(1,6,3). This will generate 3 random numbers between 1 and 6 inclusive. To roll again, just press "enter," and three new random numbers between 1 and 6 will appear.

Make a tally sheet for the sum of 3 dice. The minimum sum will be 3, and the maximum sum will be 18. Use your calculator to simulate rolling 3 dice 100 times. Record the sums on your tally sheet.

Make a frequency distribution and find the mean, mean deviation, median, and IQR.

Pool the class data on the board or the overhead calculator and make a class frequency distribution. Calculate the class mean, mean deviation, median and IQR. Discuss what these numbers represent.

Use the class distribution to answer the following probability questions:

1. $P(sum = 5) =$ ________
2. $P(sum \leq 4) =$________
3. $P(sum > 4) =$ ___________
4. $P(sum > 4 \text{ or } sum = 3) =$__________
5. What is the probability that the sum = 6 if the first die was a 3?____
6. What is the probability that the sum = 12 if the sum of the first two dice is 10?___

Calculate the theoretical probabilities of obtaining a sum of 3, 4, 5, …., 18. On the same piece of paper, construct a theoretical probability distribution. Calculate the mean, mean deviation, median and IQR and compare it to the experimental probability distribution.

Using the theoretical probabilities, compute the answers to the same questions above. Compare the answers.

7. $P(sum = 5) =$ ________
8. $P(sum \leq 4) =$________
9. $P(sum > 4) =$ ___________
10. $P(sum > 4 \text{ or } sum = 3) =$__________
11. What is the probability that the sum = 6 if the first die was a 3?____
12. What is the probability that the sum = 12 if the sum of the first two dice is 10?___

Now compute the following probabilities (if you roll 3 dice):

13. What is the probability of getting three 1's on the first roll?________
14. What is the probability of getting three of a kind on the first roll?__________
15. What is the probability of getting two 1's and another number on the first roll (in any order)?_______
16. What is the probability of getting two of a kind (3^{rd} dice must be different)?_____
17. What is the probability of getting three consecutive numbers (but in any order) on the first roll?_________

Game 1 Learning Task

#1 Work in small groups to determine if the game described below is fair. Justify.

A person pays $2 to play a game. He rolls two dice for this game. If he rolls an even sum, he wins $2.50 and goes home. If he rolls a sum of 3, 5, or 7, then he loses and goes home. If he rolls a sum of 9 or 11, he rolls again. If on his second roll, he rolls a sum of 9 or 11, he wins $5.00; otherwise, he loses and goes home.

#2: Create your own game in groups using 3 dice. Calculate the amount a person is expected to win or lose each time he plays the game. Make the game so that it's not much more likely to win as to lose.

After you have finished creating your game, compute the expected values. Play the game several times. Who won more times? Is that what you expected?

Game 2 Learning Task

Pass out the yahtzee board game for students to play. Roll 5 dice. On the first roll, record whether or not you get the following:

Only 3 of a kind:
Only 4 of a kind:
Full House (3 of a kind and 2 of a kind):
Small Straight (sequence of 4 in any order):
Large Straight (sequence of 5 in any order):
Yahtzee (5 of a kind):
None of the above:
Ask them to record their data.

Is it likely to get any of the above on the first roll? Which is most likely? How many points are awarded to this outcome? Why?

Which is the least likely outcome? How many points are awarded to this outcome? Why?

Now simulate rolling 5 dice on your calculator by entering randInt(1,6,5). Roll 5 dice 100 times each. Make a tally sheet to record the following:
Only 3 of a kind:
Only 4 of a kind:
Full House (3 of a kind and 2 of a kind):
Small Straight (sequence of 4 in any order):
Large Straight (sequence of 5 in any order):
Yahtzee (5 of a kind):
None of the above:

Pool the class data to calculate the experimental probabilities of the above outcomes.
Work in small groups to calculate the theoretical probabilities.

#2: Students should calculate the expected score on the bottom half of their yahtzee score sheet.

Game 3 Learning Task

Two players play a game. The first player rolls a pair of dice. If the sum is 6 or less, then player 1 wins. If it's more than six, then player 2 gets to roll. If player 2 gets a sum of 6 or less, then he loses. If player 2 gets a sum greater than 6, then he wins.

a) Which player, player 1 or player 2, is more likely to win? Why?
b) If player 1 is awarded 10 tokens each time he/she wins the game, how many tokens must player 2 be awarded in order for this to be a fair game? Why?

Culminating Task

A student rolled 3 dice 100 times, found the sums of the 3 dice, and put them into the following frequency distribution:

Sum of Three Dice	Frequency	Experimental Probability
3	1	
4	3	
5	5	
6	4	
7	8	
8	9	
9	13	
10	15	
11	12	
12	13	
13	7	
14	3	
15	6	
16	1	
17	2	
18	0	

a) Based on the student's simulation, compute the experimental probabilities for the sum of 3 dice and write them in the table above.

b) Based on the student's simulation, what is the expected value (the mean) of the sum of the three dice?

c) Based on the student's simulation, what is the median sum of the three dice?

d) Comment on the relationship between the mean and median relative to the shape of the distribution.

e) Based on the student's simulation, what is the probability that the sum of 3 dice is even?

Sum of Three Dice	Theoretical probability P(x)
3	1/216
4	3/216
5	6/216
6	10/216
7	15/216
8	21/216
9	25/216
10	27/216
11	27/216
12	25/216
13	21/216
14	15/216
15	10/216
16	6/216
17	3/216
18	1/216

f) Based on the theoretical probabilities in the table to the left, what is the expected value (the mean) of the sum of the three dice?

g) Based on the theoretical probabilities, what is the median sum of the three dice?

h) Comment on the relationship between the mean and median relative to the shape of the distribution.

i) Based on the theoretical probabilities, what is the probability that the sum of 3 dice is even?

j) How does the theoretical probability that the sum of 3 dice is even compare to the experimental probability that the sum of 3 dice is even (part e).

k) How does the theoretical mean and median compare to the experimental mean and median from the student's simulation?

l) Display the experimental probability distribution and the theoretical probability distribution graphically so that they can be easily compared.

m) Based on your answers to parts "j, k, and l" above, do you think that the student really simulated rolling 3 dice 100 times, or did the student make up the data. Explain.

Teachers: Your feedback regarding the tasks in this unit is welcomed. After your students have completed the tasks, please send suggestions to Janet Davis (jdavis@doe.k12.ga.us).