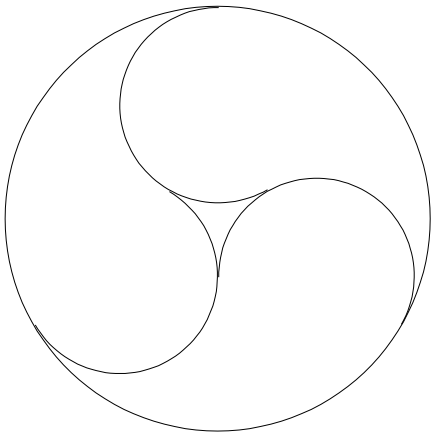
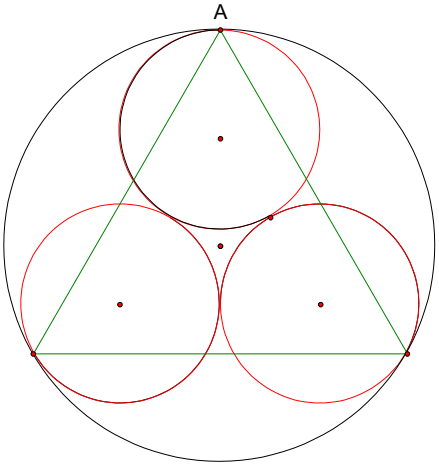
PROBLEM:

Using a straightedge and compass, construct a figure like the stained glass design shown below. What is the ratio of the area of the "triangular" region to the area of the circle?

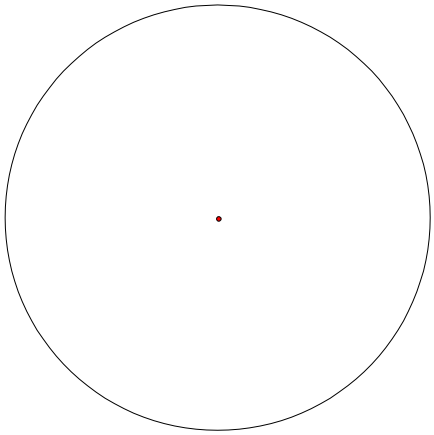


We can see three congruent tangent circles in this figure. We also know that the triangle whose vertices are the circles' points of tangency with the large circle is equilateral, since the figure has rotational symmetry. (See figure below).



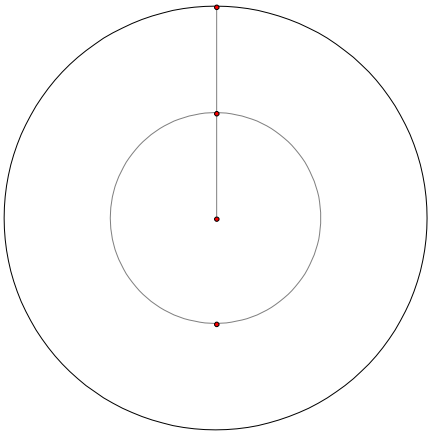
We need to construct these three circles.

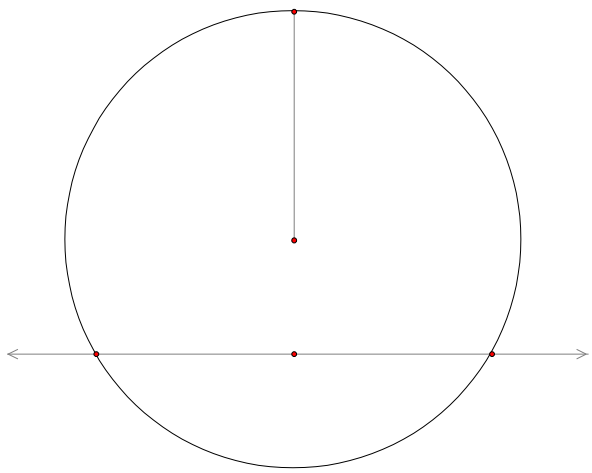
Since the window maker most likely knows what size window he's working with, let's begin with the large circle.



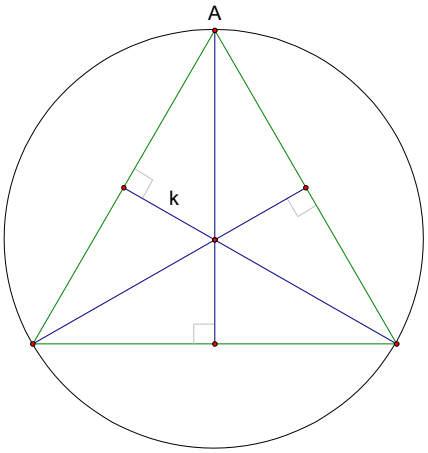
We need to inscribe an equilateral triangle in this circle (the green triangle above).

The figure has rotational symmetry about the center, so the equilateral triangle's centroid will be the center of the circle. We know a triangle's centroid is twice as far from a vertex as it is from the opposite side, so after we've chosen one point on the circle to be a vertex of the green triangle, we just need to find a point on the opposite side of the center that is half as far away from the center. (See figure at right.) This new point lies on the opposite side of the triangle; this side is perpendicular to the median we have constructed.



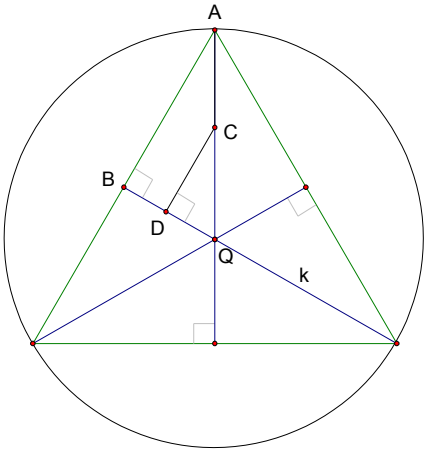


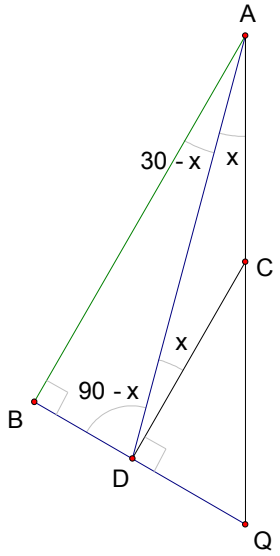
Now we have found the other two vertices of our inscribed green triangle. Let's go ahead and draw in the green triangle's altitudes.



Because the green triangle's vertices are the points of tangency with the three small circles, the centers of the three small circles must lie on these altitudes (since the altitudes go through the center of the circle—again because it is an equilateral triangle). But where?

Well, we also know that the small circles' points of tangency *with each other* lie on the altitudes (by rotational symmetry), so we need to find a point C on the altitude such that its distance from point A is equal to its distance from line k (these will both be radii of the small circle).





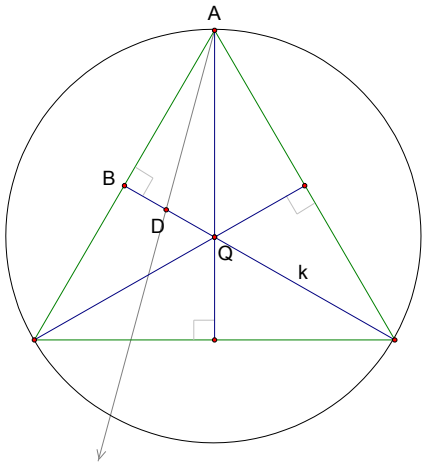
If we knew where point D was, then C would be on the perpendicular bisector of $\overline{AD}$, and ΔACD would be isosceles. Let's zoom in on just this area. (See figure at left). Since the green triangle is equilateral, we know some angles already. Let x be the base angle of ΔACD, then we can fill in other values as shown. This gives us:

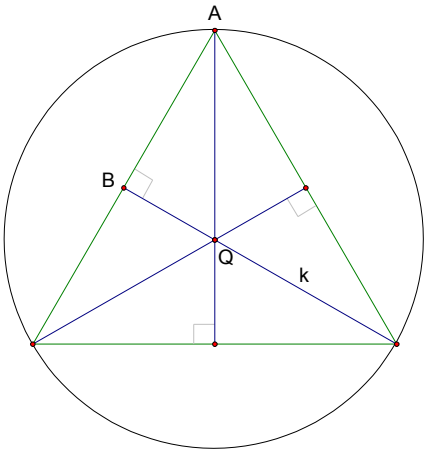
$$90^\circ + (90^\circ - x) + (30^\circ - x) = 180^\circ \quad \rightarrow \quad 210^\circ - 2x = 180^\circ$$
$$\rightarrow \quad 2x = 30^\circ \quad \rightarrow \quad x = 15^\circ$$

So $\overline{AD}$ is part of the angle bisector of ∠BAQ! And we know how to construct angle bisectors, so we're almost there.

Let's return to what we have so far, so we can continue our construction.

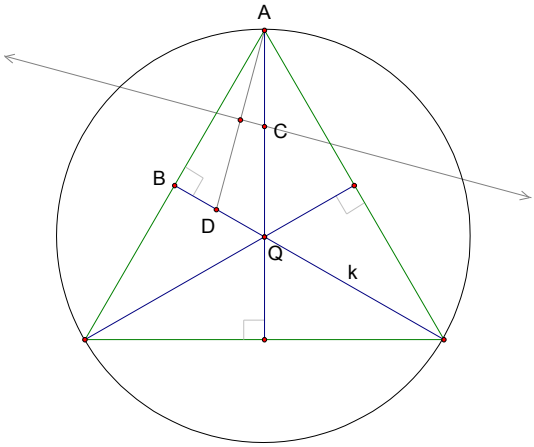
We need to construct the angle bisector of $\angle BAQ$, and then find its intersection with altitude k. Call this point D.



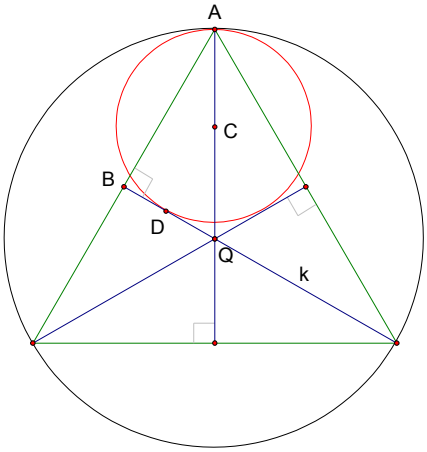


Now,

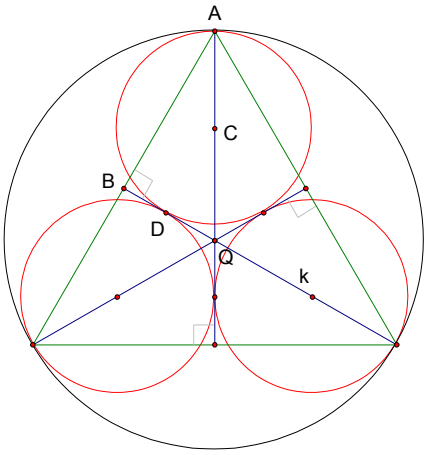
construct the perpendicular bisector of $\overline{AD}$, and label its intersection with $\overline{AQ}$ C. This is the center of our small circle.



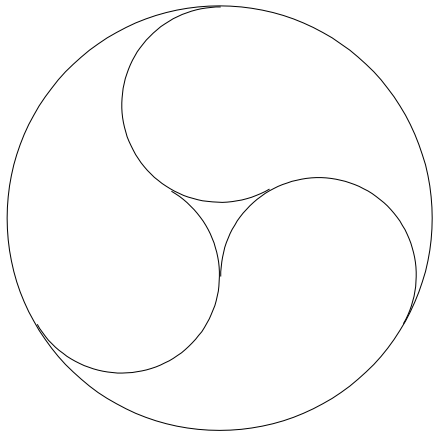
Now just draw a circle that has center C and contains point A.



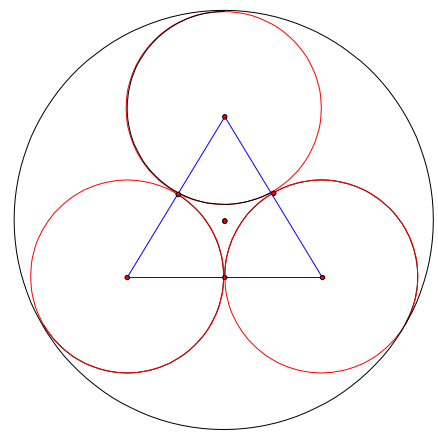
Repeat:



Now we just need to erase everything but the arcs of these circles that we want, and we have our figure:



What is the ratio of the area of the "triangular" region to the area of the entire circle? Let r be the radius of the small circles, and let R be the radius of the large circle.



If we look back at our figure before we "erased" construction lines, we can see that the triangle whose vertices are the centers of the small triangles is equilateral, with each side measuring $2r$.

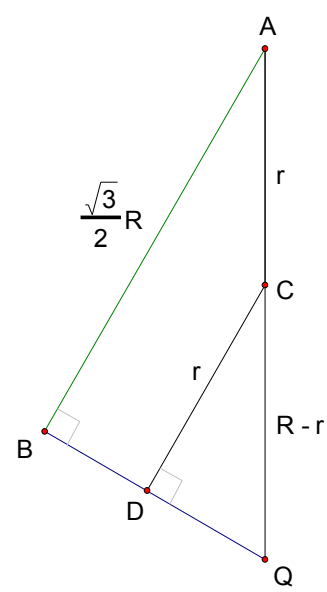
By Heron's formula, the area of this triangle is

$$\sqrt{(3x)(x)(x)(x)} = \sqrt{3x^4} = x^2\sqrt{3}$$

Now let's take out the pieces of the small circles that lie inside this triangle. Each angle measure of the triangle is 60º, so there are three sixth-circles inside. Thus, the area of the "triangular" region is

$$r^2\sqrt{3} - \left(\tfrac{1}{2}\right)(\pi r^2) = r^2(\sqrt{3} - \tfrac{\pi}{2})$$

What is r in terms of R? If we zoom in on the same area as we did before, we have the figure at right.



Now we have similar triangles to work with:

$$\frac{r}{R-r} = \frac{\frac{\sqrt{3}}{2}R}{R} = \frac{\sqrt{3}}{2} \quad \rightarrow \quad 2r = (\sqrt{3})(R-r)$$

$$\rightarrow \quad r(2+\sqrt{3}) = R\sqrt{3} \quad \rightarrow \quad \frac{r}{R} = \frac{\sqrt{3}}{2+\sqrt{3}}$$

Combining all this, we have:

$$\frac{\text{Area}_{\text{"triangle"}}}{\text{Area}_{\text{whole}}} = \frac{r^2(\sqrt{3}-\frac{\pi}{2})}{\pi R^2} = \left(\frac{r}{R}\right)^2\left(\frac{\sqrt{3}-\frac{\pi}{2}}{\pi}\right) = \left(\frac{3}{7+4\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\pi} - \frac{1}{2}\right) \approx \mathbf{0.011}$$