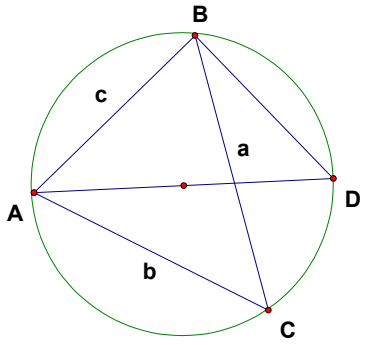
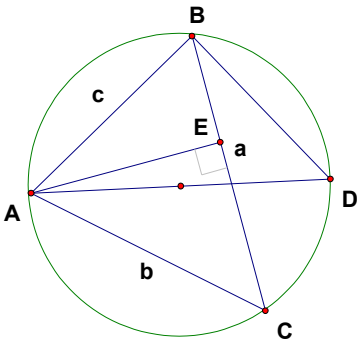
PROBLEM:

Given a triangle with side lengths a, b, and c, and a circumcircle of radius R, show that the area of the triangle is given by $\frac{abc}{4R}$.

Construct a diameter of the circumcircle through point A, and call this diameter's second intersection with the circumcircle point D. Draw in segment $\overline{BD}$ as shown at right.





Construct an altitude from A to side $\overline{BC}$. Call its intersection with $\overline{BC}$ point E. See diagram at left.

$\angle ABD$ subtends an arc of measure 180º ($\overline{AD}$ is a diameter), so $\angle ABD$ is right (see Circle Theorems Problem).

And $\angle AEC$ is right by the definition of an altitude.

Thus, $\angle ABD \cong \angle AEC$.

Also, $\angle ADB$ and $\angle ACB$ subtend the same segment (namely, $\overline{AB}$), so they are congruent as well (see Circle Theorems Problem).

By Angle-Angle similarity, $\Delta ABD \sim \Delta AEC$. Thus,

$$\frac{AC}{AE} = \frac{AD}{AB} \quad \rightarrow \quad \frac{b}{AE} = \frac{2R}{c} \quad \rightarrow \quad AE = \frac{bc}{2R}$$

$\overline{AE}$ is an altitude of ΔABC, so the area of ΔABC is

$$\frac{1}{2} \cdot a \cdot AE = \frac{1}{2} \cdot a\left(\frac{bc}{2R}\right) = \frac{abc}{4R}$$

Q.E.D.