Chapter 1

NUMERAL SYSTEMS

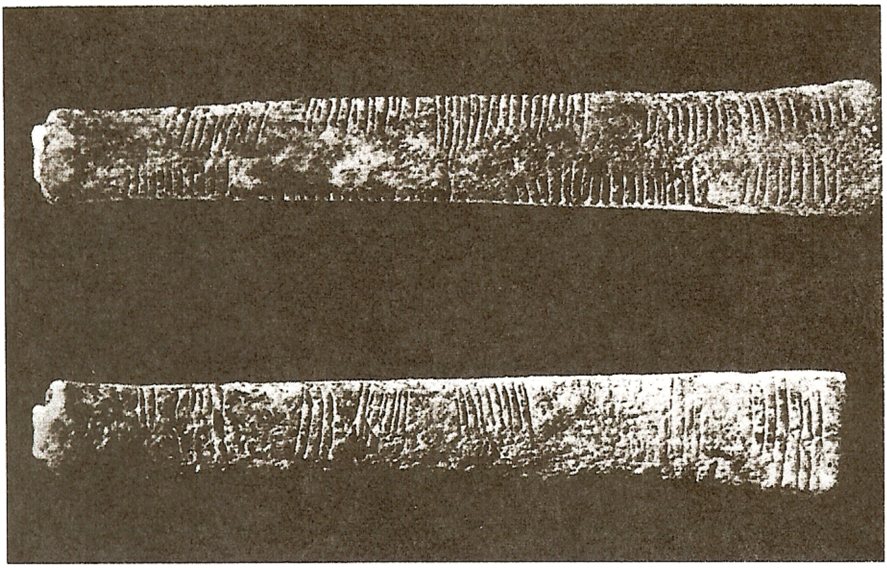
1–1 Primitive Counting

In giving a chronological account of the development of mathematics, one is beset with the problem of where to begin. Should one start with the first methodical deductions in geometry traditionally credited to Thales of Miletus around 600 B.C.? Or should one go back further and start with the empirical derivation of certain mensuration formulas made by the pre-Greek civilizations of Mesopotamia and Egypt? Or should one go back even further and start with the first groping efforts made by prehistoric man to systematize size, shape, and number? Or can one say mathematics originated in prehuman times in the meager number sense and pattern recognition of certain animals, birds, and insects? Or even before this, in the number and spatial relations of plants? Or still earlier, in the spiral nebulae, the courses of planets and comets, and the crystallization of minerals in preorganic times? Or was mathematics, as Plato believed, *always* in existence, merely awaiting discovery? Each of these possible origins can be defended.[1]

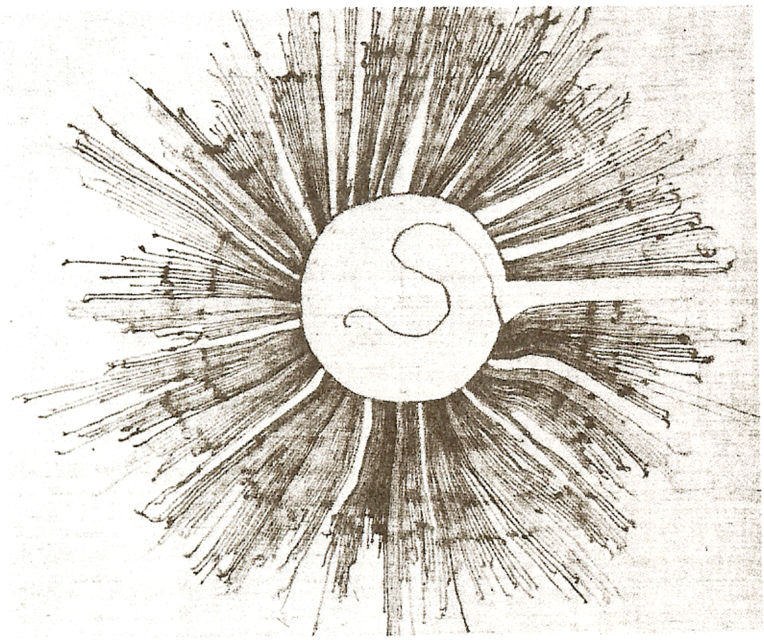
Since it is man's primal efforts to systematize the concepts of size, shape, and number that are popularly regarded as the earliest mathematics, we shall commence there, and begin with the emergence in primitive man of the concept of *number* and the process of *counting*.

The number concept and the counting process developed so long before the time of recorded history (there is archeological evidence that counting was employed by man as far back as 50,000 years ago) that the manner of this development is largely conjectural. It is not difficult, though, to imagine how it probably came about. It seems fair to argue that humans, even in most primitive times, had some number sense, at least to the extent of recognizing *more* and *less* when some objects were added to or taken from a small group, for studies have shown that some animals possess such a sense. With the gradual evolution of society, simple counting became imperative. A tribe had to know how many members it had and how many enemies, and a man found it necessary to know if his flock of sheep was decreasing in size. Probably the earliest way of keeping a count was by some simple tally method, employing the principle of one-to-one correspondence. In keeping a count on sheep, for exam-

[1] For a start, see D. E. Smith, *History of Mathematics,* vol. 1, chap. 1, and Howard Eves, *In Mathematical Circles* (Items 1°, 2°, 3°, 4°), which are cited in the General Bibliography at the end of the book.

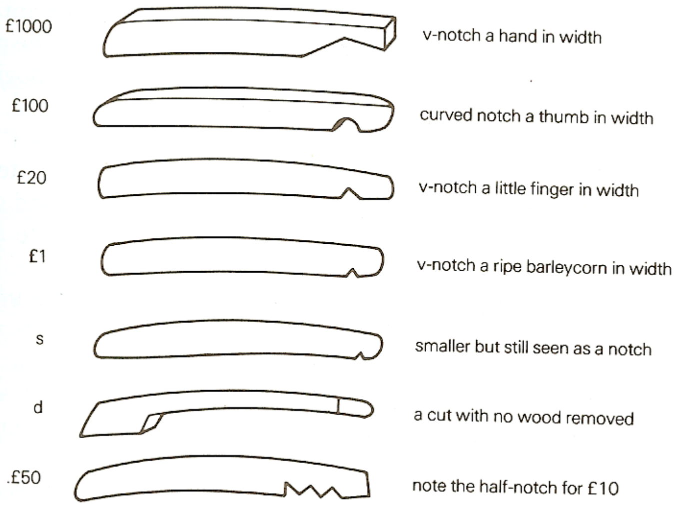


Two views of the Ishango bone, over 8000 years old and found at Ishango, on the shore of Lake Edward in Zaire (Congo), showing numbers preserved by notches cut in the bone.
(Dr. de Heinzelin.)



A Peruvian Indian census quipu, showing numbers recorded by knots in cord. Larger knots are multiples of smaller ones, and cord color may distinguish male from female.
(Collection Musée de L'Homme, Paris.)

ple, one finger per sheep could be turned under. Counts could also be maintained by making collections of pebbles or sticks, by making scratches in the dirt or on a stone, by cutting notches in a piece of wood, or by tying knots in a string. Then, perhaps later, an assortment of vocal sounds was developed as a word tally against the number of objects in a small group. And still later, with the refinement of writing, an assortment of symbols was devised to stand for these numbers. Such an imagined development is supported by reports of anthropologists in their studies of present-day primitive peoples.



Drawing showing the official system of notching used on twelfth-century exchequer tallies of the British Royal Treasury. Such tallies continued in use until 1826.

In the earlier stages of the period of vocal counting, different sounds (words) were used, for example, for *two* sheep and *two* men. (Consider, for example, in English: *team* of horses, *span* of mules, *yoke* of oxen, *brace* of partridge, *pair* of shoes, *couple* of days.) The abstraction of the common property of *two*, represented by some sound considered independently of any concrete association, probably was a long time in arriving. Our present number words in all likelihood originally referred to sets of certain concrete objects, but these associations, except for that perhaps relating five and hand, are now lost to us.[2]

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[2] For an interesting alternative to the classical evolutionary view of nonliterate peoples, see Marcia and Robert Ascher, "Euthomathematics," *History of Science* 24, no. 2 (June 1980): 125–144.

1–2 Number Bases

When it became necessary to make more extensive counts, the counting process had to be systematized. This was done by arranging the numbers into convenient basic groups, the size of the groups being largely determined by the matching process employed. Essentially, the method was like this. Some number b was selected as a base (also called **radix** or **scale**) for counting, and names were assigned to the numbers $1, 2, \ldots, b$. Names for numbers larger than b were then given by combinations of the number names already selected.

Since fingers furnished such a convenient matching device, it is not surprising that 10 was ultimately chosen far more often than not for the number base b. Consider, for example, our present number words, which are formed on 10 as a base. We have the special names *one, two, . . . , ten* for the numbers 1, 2, . . . , 10. When we come to 11, we say *eleven,* which, the philologists tell us, derives from *ein lifon,* meaning "one left over," or one over ten. Similarly, *twelve* is from *twe lif* ("two over ten"). Then we have *thirteen* ("three and ten"), *fourteen* ("four and ten"), up through *nineteen* ("nine and ten"). Then comes *twenty* (*twe-tig*, or "two tens"), *twenty-one* ("two tens and one"), and so on. The word *hundred,* we are told, comes originally from a term meaning "ten times" (ten).

There is evidence that 2, 3, and 4 have served as primitive number bases. For example, there are natives of Queensland who count "one, two, two and one, two twos, much," and some African pygmies count "*a, oa, ua, oa-oa, oa-oa-a,* and *oa-oa-oa*" for 1, 2, 3, 4, 5, and 6. A certain tribe of Tierra del Fuego has its first few number names based on 3, and some South American tribes similarly use 4.

As might be expected, the **quinary scale,** or number system based on 5, was the first scale to be used extensively. To this day, some South American tribes count by hands: "one, two, three, four, hand, hand and one," and so on. The Yukaghirs of Siberia use a mixed scale by counting "one, two, three, three and one, five, two threes, one more, two three-and-ones, ten with one missing, ten." German peasant calendars used a quinary scale as late as 1800.

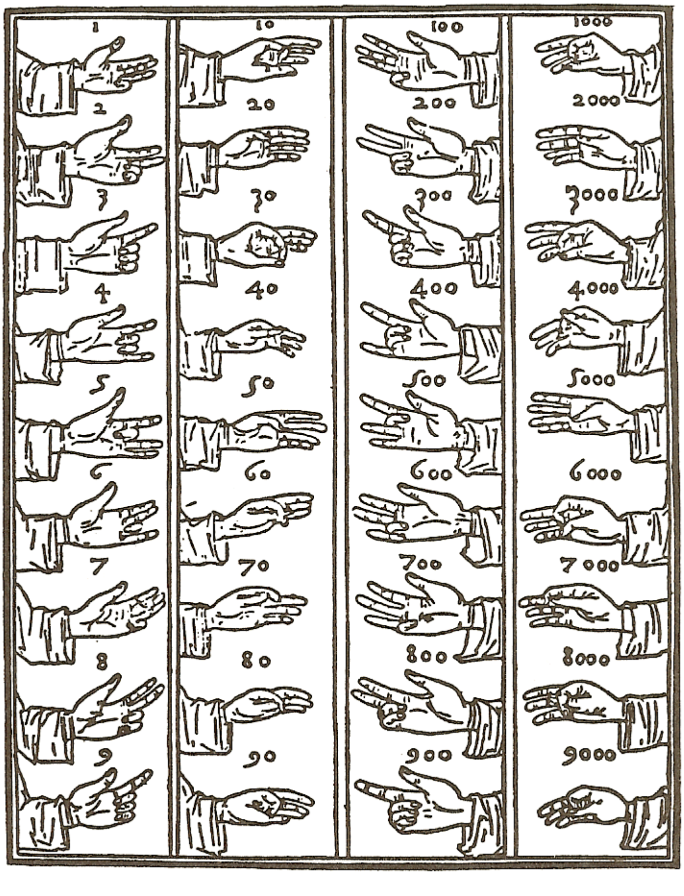
There is evidence that the **duodecimal scale,** or number system based on 12, may have been used in some societies during prehistoric times, chiefly in relation to measurements. Such a base may have been suggested by the approximate number of lunations in a year, or perhaps because 12 has so many integral fractional parts. At any rate, we have 12 as the number of inches in a foot, ounces in the ancient pound, pence in a shilling, lines in an inch, hours about the clock, months in a year, and the words *dozen* and *gross* used as higher units.

The **vigesimal scale,** or number system based on 20, has been widely used, and recalls man's barefoot days. This scale was used by American Indian peoples, and is best known in the well-developed Mayan number system. Celtic traces of a base 20 are found in the French *quatre-vingt* instead of *huitante,* and *quatre-vingt-dix* instead of *nonante*. Traces are also found in Gaelic, Danish, and Welsh. The Greenlanders use "one man" for 20, "two men" for 40, and so on. In English we have the frequently used word *score*.

The **sexagesimal scale,** or number system based on 60, was used by the ancient Babylonians, and is still used when measuring time and angles in minutes and seconds.

1–3 Finger Numbers and Written Numbers

In addition to spoken numbers, **finger numbers** were at one time widely used. Indeed, the expression of numbers by various positions of the fingers and hands probably predates the use of either number symbols or number names. Thus, the early written symbols for 1, 2, 3, and 4 were invariably the suitable number of vertical or horizontal strokes, representing the corresponding number of raised or extended fingers, and the word *digit* (that is, "finger") for the numbers 1 through 9 can be traced to the same source.



Finger numbers from Pacioli's *Sūma* of 1494. The first two columns represent the left hand, the other two the right hand.

In time, finger numbers were extended to include the largest numbers occurring in commercial transactions; by the Middle Ages, they had become international. In the ultimate development, the numbers 1, 2, . . . , 9 and 10, 20, . . . , 90 were represented on the left hand, and the numbers 100, 200, . . . , 900 and 1000, 2000, . . . , 9000 on the right hand. In this way, any number up to 10,000 was representable by the use of the two hands. Pictures of the finger numbers were given in Renaissance arithmetic books. For example, using the left hand, 1 was represented by partially folding down the little finger; 2 by partially folding down the little and ring fingers; 3 by partially folding down the little, ring, and middle fingers; 4 by folding down the middle and ring fingers; 5 by folding down the middle finger; 6 by folding down the ring finger; 7 by completely folding down the little finger; 8 by completely folding down the little and ring fingers; and 9 by completely folding down the little, ring, and middle fingers.

Although finger numbers originated in very early times, they are still used today by some primitive races of Africa, by Arabs, and by Persians. In North and South America, some native Indian and Eskimo tribes still employ the fingers.

Finger numbers had the advantage of transcending language differences but, like the vocal numbers, lacked permanence and were not suitable for performing calculations. We have already mentioned the use of marks and notches as early ways of recording numbers. In such devices, we probably have the first attempt at writing. At any rate, various written number systems gradually evolved from these primitive efforts to make permanent number records. A written number is called a **numeral,** and we now turn our attention to a simple classification of early numeral systems.

1–4 Simple Grouping Systems

Perhaps the earliest type of numeral system that was developed is that which has been called a **simple grouping system.** In such a system, some number b is selected for number base, and symbols are adopted for 1, b, b^2, b^3, and so on. Then any number is expressed by using these symbols *additively,* each symbol being repeated the required number of times. The following illustrations will clarify the underlying principle.

A very early example of a simple grouping system is that furnished by the Egyptian hieroglyphics, employed as far back as 3400 B.C. and chiefly used by the Egyptians when making inscriptions on stone. Although the hieroglyphics were sometimes used on other writing media than stone, the Egyptians early developed two considerably more rapid writing forms for work on papyrus, wood, and pottery. The earlier of these forms was a running script, known as the **hieratic,** derived from the hieroglyphic and used by the priesthood. From the hieratic, there later evolved the **demotic** writing, which was adopted for general use. The hieratic and demotic numeral systems are not of the simple grouping type.

The Egyptian hieroglyphic numeral system is based on the scale of 10. The symbols adopted for 1 and the first few powers of 10 are

1 a vertical staff, or stroke

10  a heel bone, or hobble, or yoke

10^2  a scroll, or coil of rope

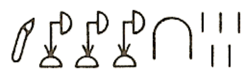
10^3  a lotus flower

10^4  a pointing finger

10^5  a burbot fish, or tadpole

10^6  a man in astonishment, or a god holding up the universe

Any number is now expressed by using these symbols additively, each symbol being repeated the required number of times. Thus,

$$13015 = 1(10^4) + 3(10^3) + 1(10) + 5 =$$ 

We have written this number from left to right, although it was more customary for the Egyptians to write from right to left.

The early Babylonians, lacking papyrus and having little access to suitable stone, resorted principally to clay as a writing medium. The inscription was pressed into a wet clay tablet by a stylus, the writing end of which may have been a sharp isosceles triangle. By tilting the stylus slightly from the perpendicular, one could press either the vertex angle or a base angle of the isosceles triangle into the clay, producing two forms of wedge-shaped **(cuneiform)** characters. The finished tablet was then baked in an oven to a time-resisting hardness that resulted in a permanent record. On cuneiform tablets dating from 2000 to 200 B.C., numbers less than 60 are expressed by a simple grouping system to base 10, and it is interesting that the writing is often simplified by using a subtractive symbol. The subtractive symbol and the symbols for 1 and 10 are

respectively, where the symbol for 1 and the two parts making up the subtractive symbol are obtained by using the vertex angle of the isosceles triangle, and the symbol for 10 is obtained by using one of the base angles. As examples of written numbers employing these symbols, we have

$$25 = 2(10) + 5 =$$ 

and

$$38 = 40 - 2 =$$ 

The method employed by the Babylonians for writing larger numbers will be considered in Section 1-7.

The Attic, or Herodianic, Greek numerals were developed some time prior to the third century B.C. and constitute a simple grouping system to base 10 formed from initial letters of number names. In addition to the symbols I, Δ, H, X, M for 1, 10, 10^2, 10^3, 10^4, there is a special symbol for 5. This special symbol is an old form of Π, the initial of the Greek *pente* ("five"), and Δ, H, X, and M are the initial letters of the Greek *deka* (ten), *hekaton* (hundred), *kilo* (thousand), and *myriad* (ten thousand). The symbol for 5 was frequently used both alone and in combination with other symbols in order to shorten number representations. As an example, in this numeral system we have

$$2857 =$$ 

in which one can note the special symbol for 5 appearing once alone and twice in combination with other symbols.

As a final example of a simple grouping system, again to base 10, we have the familiar Roman numerals. Here the basic symbols I, X, C, M for 1, 10, 10^2, 10^3 are augmented by V, L, D for 5, 50, and 500. The subtractive principle, in which a symbol for a smaller unit placed before a symbol for a larger unit means the difference of the two units, was used only sparingly in ancient and medieval times. The fuller use of this principle was introduced in modern times. As an example, in this system we have

$$1944 = \text{MDCCCCXXXXIIII},$$

or, in more modern times, when the subtractive principle became common,

$$1944 = \text{MCMXLIV}.$$

In using the subtractive principle, however, one is to abide by the following rule: I can precede only V or X, X can precede only L or C, C can precede only D or M.

There has been no lack of imagination in the attempts to account for the origins of the Roman number symbols. Among the more plausible explanations, acceptable to many authorities on Latin history and epigraphy, is that I, II, III, IIII were derived from the raised fingers of the hand. The symbol X may be a compound of two V's, or may have been suggested by crossed hands or thumbs, or may have originated from the common practice of crossing groups of ten when counting by strokes. There is some evidence that the original symbols for 50, 100, and 1000 may have been the Greek aspirates Ψ (*psi*), θ (*theta*), and Φ (*phi*). Older forms for *psi* were

$$\downarrow, \perp, \perp, \llcorner,$$

all of which were used for 50 in early inscriptions. The symbol θ for 100 probably later developed into the somewhat similar symbol C, influenced by the fact that C is the initial letter of the Latin word *centum* ("hundred"). A commonly used early symbol for 1000 is C|Ɔ, which could be a variant of Φ. The symbol for 1000 became an M, influenced by the fact that M is the initial letter of the Latin word *mille* ("thousand"). Five hundred, being half of 1000, was represented by |Ɔ, which later became a D. The symbols C|Ɔ and |Ɔ for 1000 and 500 are found as late as 1715.

1–5 Multiplicative Grouping Systems

There are instances in which a simple grouping system developed into what may be called a **multiplicative grouping system.** In such a system, after a base b has been selected, symbols are adopted for $1, 2, \ldots, b - 1$, and a second set of symbols for $b, b^2, b^3, \ldots$. The symbols of the two sets are employed *multiplicatively* to show how many units of the higher groups are needed. Thus, if we should designate the first nine numbers by the usual symbols, but designate 10, 100, and 1000 by a, b, c, say, then in a multiplicative grouping system we would write

$$5625 = 5c6b2a5.$$

The traditional Chinese-Japanese numeral system is a multiplicative grouping system to base 10. Writing vertically, the symbols of the two basic groups and of the number 5625 are as shown on p. 18.

Lacking a paperlike writing material, the early Chinese and Japanese recorded their findings on bamboo slips. The piece of a bamboo stalk between two knots was split lengthwise into thin strips. After these strips were dried and scraped, they were laid side by side and tied together by four crosswise cords. The narrowness of the strips necessitated that the characters written on them be arranged vertically from top to bottom, giving rise to a custom of writing that persisted into more modern times, when bamboo slips were replaced by silk and paper as more convenient writing materials.

				Example: 5625
1	一	10	十	五
2	二	10^2	百	千
3	三	10^3	千	六
4	四			百
5	五			二
6	六			十
7	七			五
8	八			
9	九			

1–6 Ciphered Numeral Systems

In a **ciphered numeral system,** after a base b has been selected, sets of symbols are adopted for $1, 2, \ldots, b-1$; $b, 2b, \ldots, (b-1)b$; $b^2, 2b^2, \ldots, (b-1)b^2$; and so on. Although many symbols must be memorized in such a system, the representation of numbers is compact.

The so-called Ionic, or alphabetic, Greek numeral system is of the ciphered type and can be traced as far back as about 450 B.C. It is a system that is based on 10 and employs twenty-seven characters—the twenty-four letters of the Greek alphabet together with the symbols for the obsolete *digamma, koppa,* and *sampi*. Although the capital letters were used (the small letters were substituted much later), we shall now illustrate the system with the small letters. The following equivalents had to be memorized:

1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau
4	δ	delta	40	μ	mu	400	υ	upsilon
5	ε	epsilon	50	ν	nu	500	ϕ	phi
6	obsolete	digamma	60	ξ	xi	600	χ	chi
7	ζ	zeta	70	o	omicron	700	ψ	psi
8	η	eta	80	π	pi	800	ω	omega
9	θ	theta	90	obsolete	koppa	900	obsolete	sampi

As examples of the use of these symbols, we have

$$12 = \iota\beta, \qquad 21 = \kappa\alpha, \qquad 247 = \sigma\mu\zeta.$$

Accompanying bars or accents were used for larger numbers (see Problem Study 1.3 (b)).

Symbols for the obsolete digamma, koppa, and sampi are

ϛ, ϙ, ϡ.

Other ciphered numeral systems are the Egyptian hieratic and demotic, Coptic, Hindu Brahmi, Hebrew, Syrian, and early Arabic. The last three, like the Ionic Greek, are *alphabetic* ciphered numeral systems.

1–7 Positional Numeral Systems

Our own numeral system is an example of a **positional numeral system** with base 10. For such a system, after the base b has been selected, basic symbols are adopted for $0, 1, 2, \ldots, b-1$. Thus, there are b basic symbols, frequently called the **digits** of the system. Now any (whole) number N can be written uniquely in the form

$$N = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_2 b^2 + a_1 b + a_0,$$

where $0 \leqq a_i < b$, $i = 0, 1, \ldots, n$. We then represent the number N to base b by the sequence of basic symbols

$$a_n a_{n-1} \ldots a_2 a_1 a_0.$$

Thus, a basic symbol in any given numeral represents a multiple of some power of the base, the power depending on the position in which the basic symbol occurs. In our own *Hindu-Arabic* numeral system, for example, the 2 in 206 stands for $2(10^2)$, or 200, whereas in 27, the 2 stands for 2(10), or 20. Note that for complete clarity some symbol for zero is needed to indicate any possible missing powers of the base. A positional numeral system is a logical, although not necessarily historical, outgrowth of a multiplicative grouping system.

Sometime between 3000 and 2000 B.C., the ancient Babylonians evolved a sexagesimal system employing the principle of position. The numeral system, however, is really a mixed one in that, although numbers exceeding 60 are written according to the positional principle, numbers within the basic 60 group are written by a simple grouping system to base 10, as explained in Section 1–4. As an illustration we have

$$524{,}551 = 2(60^3) + 25(60^2) + 42(60) + 31 =$$ 

Until after 300 B.C., this positional numeral system suffered from the lack of a zero symbol to stand for any missing powers of 60, thus leading to possible misinterpretations of given number expressions. The symbol that was finally introduced consisted of two small, slanted wedges, but this symbol was used only to indicate a missing power of the base 60 *within* a number, and not for any missing power of the base 60 occurring at the *end* of a number. Thus, the symbol was only a partial *zero*, for a true zero serves for missing powers of the base both within and at the end of numbers, as in our 304 and 340. In the Babylonian numeral system, then, 10,804 would appear as

$$10{,}804 = 3(60^2) + 0(60) + 4 =$$ 

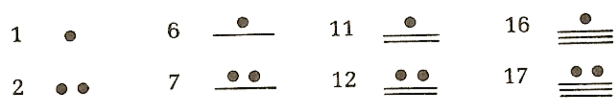
and 11,040 as

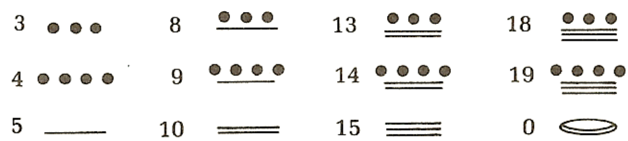
$$11{,}040 = 3(60^2) + 4(60) =$$ 

rather than as

 .

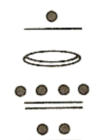
The Mayan numeral system is very interesting. Of remote but unknown date of origin, it was uncovered by the early sixteenth-century Spanish expeditions into Yucatán. This system is essentially a vigesimal one, except that the second number group is $(18)(20) = 360$ instead of $20^2 = 400$. The higher groups are of the form $(18)(20^n)$. The explanation of this discrepancy probably lies in the fact that the official Mayan year consisted of 360 days. The symbol for zero given in the table below, or some variant of this symbol, is consistently used. The numbers within the basic 20 group are written very simply by dots and dashes (pebbles and sticks) according to the following simple grouping scheme, the dot representing 1 and the dash 5.





An example of a larger number, written in the vertical Mayan manner, is shown below.

$$43{,}487 = 6(18)(20^2) + 0(18)(20) + 14(20) + 7 =$$



The mixed-base system we have described was used by the priest class. There are reports of a pure vigesimal system that was used by the common people but which has not survived in written form.

1–8 Early Computing

Many of the computing patterns used today in elementary arithmetic, such as those for performing long multiplications and divisions, were developed as late as the fifteenth century. Two reasons are usually advanced to account for this tardy development; namely, the mental difficulties and the physical difficulties encountered in such work.

The first reason, mental difficulties, must be somewhat discounted. The impression that the ancient numeral systems are not amenable to even the simplest calculations is largely based on lack of familiarity with these systems. It is clear that addition and subtraction in a simple grouping system require only the ability to count the number symbols of each kind and then to convert to higher units. No memorization of number combinations is needed.[3] In a ciphered numeral system, if sufficient addition and multiplication tables have been memorized, the work can proceed much as we do it today. The French mathematician Paul Tannery attained considerable skill in multiplication with the Greek Ionic numeral system and even concluded that that system has some advantages over our present one.

The physical difficulties encountered, however, were quite real. Without a plentiful and convenient supply of some suitable writing medium, any very extended development of arithmetic processes was bound to be hampered. It must be remembered that our common machine-made pulp paper is little more than a hundred years old. The older rag paper was made by hand; conse-

[3] For the performance of long multiplications and divisions with Roman numerals, see, for example, James G. Kennedy, "Arithmetic with Roman numerals," *The American Mathematical Monthly* 88 (1981): 29–33.

quently, it was expensive and scarce. It was not introduced into Europe until the twelfth century, although it is likely that the Chinese knew how to make it a thousand years before.

An early paperlike writing material, called **papyrus,** was invented by the ancient Egyptians, and by 650 B.C. had been introduced into Greece. It was made from a water reed called *papu,* which is found in abundance in the Nile delta. The stems of the reed were cut into long, thin strips and laid side by side to form a sheet. Another layer of strips was laid crosswise on top and the whole soaked with water, after which the sheet was pressed out and dried in the sun. Probably because of a natural gum in the plant, the layers stuck together. After the sheets were dry, they were readied for writing by laboriously smoothing them with a hard, round object. Papyrus was too valuable to be used in any quantity as mere scratch paper.

Another early writing medium was parchment, which was made from the skins of animals, usually sheep and lambs. Naturally, this was scarce and hard to get. Even more valuable was vellum, a parchment made from the skin of calves. In fact, parchment was so costly that the custom arose in the Middle Ages of washing the ink off old parchment manuscripts and using them over again. Such manuscripts are called **palimpsests** (*palin,* "again"; *psao,* "rub smooth"). In some instances, after the passage of years, the original writing of a palimpsest reappeared faintly beneath the later treatment. Some interesting restorations have been made in this manner.

Small boards bearing a thin coat of wax, along with a stylus, formed a writing medium for the Romans of about 2000 years ago. Before and during the Roman Empire, sand trays were frequently used for simple counting and for the drawing of geometrical figures. Of course, stone and clay were used very early for making written records.

The way around these mental and physical difficulties was the invention of the **abacus** (Greek *abax,* "sand tray"), which can be called the earliest mechanical computing device used by man. It appeared in many forms in parts of the ancient and medieval world. Let us describe a rudimentary form of abacus and illustrate its use in the addition and subtraction of some Roman numbers. Draw four vertical parallel lines and label them from left to right by M, C, X, and I, and obtain a collection of convenient counters, like checkers, pennies, or pebbles. A counter will represent 1, 10, 100, or 1000 units according to its position on the I, X, C, or M line. To reduce the number of counters that may subsequently appear on a line, we agree to replace any five counters on a line by one counter in the space just to the left of that line. Any number less than 10,000 may then be represented on our frame of lines by placing not more than four counters on any line, and not more than one counter in the space just to the left of that line.

Let us now add

MDCCLXIX and MXXXVII.

Represent the first of the two numbers by counters on the frame, as illustrated at the left in Figure 1. We now proceed to add the second number, working

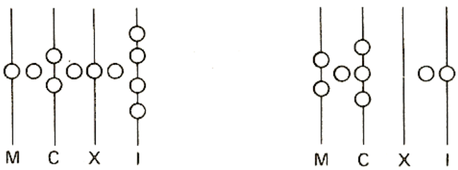


FIGURE 1

from right to left. To add the VII, put another counter between the X and I lines and two more counters on the I line. The I line now has six counters on it. We remove five of them and instead put another counter between the X and I lines. Of the three counters now between the X and I lines, we "carry over" two of them as a single counter on the X line. We now add the XXX by putting three more counters on the X line. Since we now have a total of five counters on the X line, they are replaced by a single counter between the C and X lines, and the two counters now found there are "carried over" as a single counter on the C line. We finally add the M by putting another counter on the M line. The final appearance of our frame is illustrated at the right in Figure 1, and the sum can be read off as MMDCCCVI. We have obtained the sum of the two numbers by simple mechanical operations and without requiring any scratch paper or recourse to memorization of any addition tables.

Subtraction is similarly carried out, except that now, instead of "carrying over" *to* the left, we may find it necessary to "borrow" *from* the left.

The Hindu-Arabic positional numeral system represents a number very simply by recording in order the number of counters belonging to the various lines of the abacus. The symbol 0 stands for a line with no counters on it. Our present addition and subtraction patterns, along with the concepts of "carrying over" and "borrowing" may have originated in the processes for carrying out these operations on the abacus. With the Hindu-Arabic numeral system, we are working with symbols instead of the actual counters, so it becomes necessary either to commit the simple number combinations to memory or to have recourse to an elementary addition table.

1–9 The Hindu-Arabic Numeral System

The Hindu-Arabic numeral system is named after the Hindus, who may have invented it, and after the Arabs, who transmitted it to western Europe. The earliest preserved examples of our present number symbols are found on some stone columns erected in India about 250 B.C. by King Aśoka. Other early examples in India, if correctly interpreted, are found among records cut about 100 B.C. on the walls of a cave in a hill near Poona and in some inscriptions of about A.D. 200 carved in the caves at Nasik. These early specimens contain no zero and do not employ positional notation. Positional value and a zero must have been introduced in India sometime before A.D. 800, because the Persian mathematician al-Khowârizmî describes such a completed Hindu system in a book of A.D. 825.

interests of daily life and to range over many periods of Babylonian history. There are mathematical texts dating from the latest Sumerian period of perhaps 2100 B.C.; a second and very large group from the succeeding First Babylonian Dynasty of King Hammurabi's era, and on down to about 1600 B.C.; and a third generous group running from about 600 B.C. to A.D. 300, covering the New Babylonian Empire of Nebuchadnezzar and the following Persian and Seleucidan eras. The lacuna between the second and third groups coincides with an especially turbulent period of Babylonian history. Most of our knowledge of the contents of these mathematical tablets does not predate 1935 and is largely due to the remarkable findings of Otto Neugebauer and F. Thureau-Dangin. Since the work of interpreting these tablets is still proceeding, new and perhaps equally remarkable discoveries are quite probable in the near future.

2–3 Commercial and Agrarian Mathematics

Even the oldest tablets show a high level of computational ability and make it clear that the sexagesimal positional system was already long established. There are many texts of this early period dealing with farm deliveries and with arithmetical calculations based on these transactions. The tablets show that the ancient Sumerians were familiar with all kinds of legal and domestic contracts, like bills, receipts, promissory notes, accounts, both simple and compound interest, mortgages, deeds of sale, and guaranties. There are tablets that are records of business firms, and others that deal with systems of weights and measures.

Many arithmetic processes were carried out with the aid of various tables. Of the 400 mathematical tablets, a good half contain mathematical tables. These table tablets show multiplication tables, tables of reciprocals, tables of squares and cubes, and even tables of exponentials. These latter tables were probably used, along with interpolation, for problems on compound interest. The reciprocal tables were used to reduce division to multiplication.

The calendar used by the Babylonians was established ages earlier, as evidenced by the facts that their year started with the vernal equinox and that the first month was named after Taurus. Because the sun was in Taurus at this equinox around 4700 B.C., it seems safe to say that the Babylonians had some kind of arithmetic as far back as the fourth or fifth millennium B.C.

For examples concerning Babylonian table construction and Babylonian use of tables in business transactions, see Problem Studies 2.1 and 2.2.

2–4 Geometry

Babylonian geometry is intimately related to practical mensuration. From numerous concrete examples, the Babylonians of 2000 to 1600 B.C. must have been familiar with the general rules for the area of a rectangle, the areas of right and isosceles triangles (and perhaps the general triangle), the area of a trapezoid having one side perpendicular to the parallel sides, the volume of a rectangular parallelepiped, and, more generally, the volume of a right prism with a special trapezoidal base. The circumference of a circle was taken as three times the diameter and the area as one-twelfth the square of the circumference (both correct for $\pi = 3$), and the volume of a right circular cylinder was then obtained by finding the product of the base and the altitude. The volume of a frustum of a cone or of a square pyramid is incorrectly given as the product of the altitude and half the sum of the bases. The Babylonians also knew that corresponding sides of two similar right triangles are proportional, that the perpendicular through the vertex of an isosceles triangle bisects the base, and that an angle inscribed in a semicircle is a right angle. The Pythagorean theorem was also known. (In this connection, see Section 2–6.) There is a recently discovered tablet in which $3\frac{1}{8}$ is used as an estimate for π [see Problem Study 2.5(b)].

The chief feature of Babylonian geometry is its algebraic character. The more intricate problems that are expressed in geometric terminology are essentially nontrivial algebra problems. Typical examples may be found in Problem Studies 2.3 and 2.4. There are many problems concerning a transversal parallel to a side of a right triangle that lead to quadratic equations; there are others that lead to systems of simultaneous equations, one instance giving ten equations in ten unknowns. There is a Yale tablet, possibly from 1600 B.C., in which a general cubic equation arises in a discussion of volumes of frustrums of a pyramid, as the result of eliminating z from a system of equations of the type

$$z(x^2 + y^2) = A, \qquad z = ay + b, \qquad x = c.$$

We undoubtedly owe to the ancient Babylonians our present division of the circumference of a circle into 360 equal parts. Several explanations have been put forward to account for the choice of this number, but perhaps none is more plausible than the following, advocated by Otto Neugebauer. In early Sumerian times, there existed a large distance unit, a sort of *Babylonian mile,* equal to about seven of our miles. Since the Babylonian mile was used for measuring longer distances, it was natural that it should also become a time unit—namely, the time required to travel a Babylonian mile. Later, sometime in the first millennium B.C., when Babylonian astronomy reached the stage in which systematic records of celestial phenomena were kept, the Babylonian time-mile was adopted for measuring spans of time. Since a complete day was found to be equal to twelve time-miles, and one complete day is equivalent to one revolution of the sky, a complete circuit was divided into twelve equal parts. For convenience, however, the Babylonian mile was subdivided into thirty equal parts; thus, we arrive at (12)(30) = 360 equal parts in a complete circuit.

2–5 Algebra

By 2000 B.C. Babylonian arithmetic had evolved into a well-developed rhetorical, or prose, algebra. Not only were quadratic equations solved, both by the equivalent of substituting in a general formula and by completing the square,