The University of Georgia

Mathematics Education Program
J. Wilson, EMAT 6690

The Circle of Apollonius

By

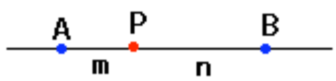
BJ Kim

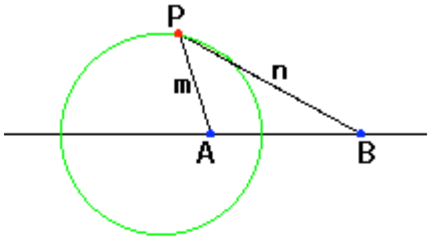
Who is Apollonius? (History of mathematician)

Apollonius of Perga (about 262 B.C- about 190 B.C.) was a Greek mathematician known as 'The Great Geometer'. His works had a very great influence on the development of mathematics and his famous book Conics introduced the terms parabola, ellipse and hyperbola.
http://www.britannica.com/EBchecked/topic/30058/Apollonius-of-Perga

What is the Circle of Apollonius?

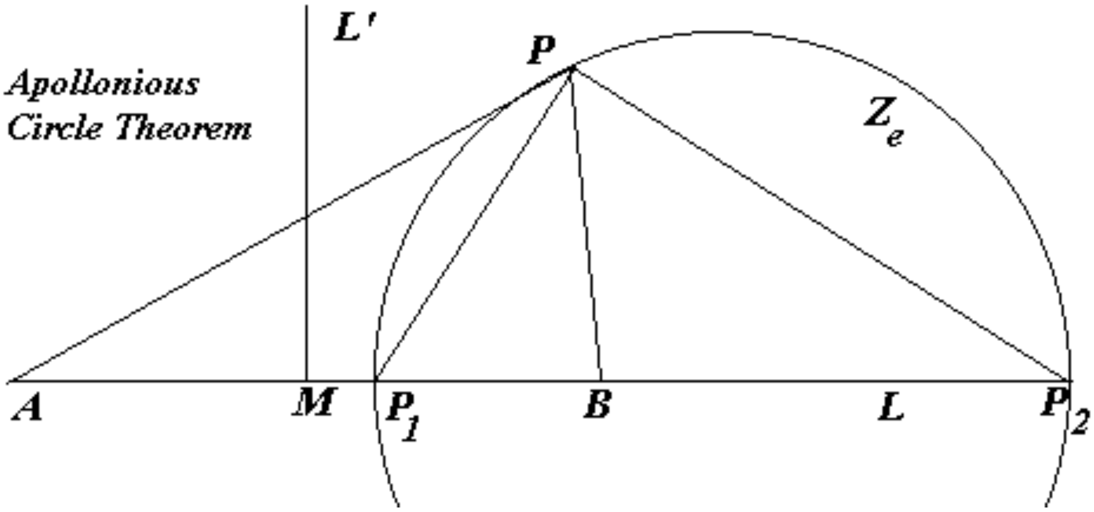
The locus of a point P whose distance from a fixed point A is a multiple of its distance from another fixed point B. If the multiple is equal to 1, then the locus is a line-- the perpendicular bisector of the segment AB. If the multiple is not equal to 1, then it is a circle. The locus is called the Circle of Apollonius.





In the diagram, AP:BP = m:n. When the point P moves keeping this ratio the locus of P is a circle. We call it the circle of Apollonius.

This circle connects interior and exterior division points of A and B.



The locus of a variable point whose distances from two fixed points are at a constant ratio k, is a circle for k≠ 1 and the perpendicular bisector of the two points for k = 1. The family of such loci for all real values of e forms a coaxial family of circles with the two fixed points as limit circles.

Proof

Let L be the line through the two fixed points A, B.

Let P be a variable point such that PA/PB = k is a constant. WLOG [without loss of generality] assume k > 1. Let P1 and P2 be two points on L, P1 between A and B, and P2 outside, such that PiA/PiB = k for i = 1, 2.

Then since PA/PB = P1A/P1B = area (ΔPAP1) / area (ΔPAP2), P1 is equidistant from PA and PB, so that PP1 bisects <APB internally. Similarly, P2 is equidistant from lines PA and PB, hence PP2 bisects <APB externally. ∴PP1 ⊥ PP2. The internal and external bisectors of at an angle of a triangle are perpendicular. The sum of the internal angle and the external angle is 180 degrees. The angle between the internal and external bisectors is the sum of one-half of each. Therefore the angle measures 90 degrees. ∴P lies on the circle Ze on P1P2 as diameter.

Let M be the midpoint of AB. ∴MP1/MB = (½) (AP1-BP1) / (½) (AP1 + BP1) = (k-1)/(k+1) = (½) (AP2-BP2) / (½) (AP2 + BP2) = MB/MP2.

∴MB2 = MP1 × MP2. ∴The tangent from M to Ze has length MB, or AB/2, which is independent of k. So for each real value of k, the different circles Ze all have L as diameter and have tangents from M having AB/2 as length, thus forming a radical family of circles with the perpendicular bisector L' of AB as radical axis. The points A and B corresponds to the cases when k = 0 and ∞ and are null circles. For k = 1, the locus is the perpendicular bisector L' of AB.

The proof of this result is based on the following theorems:

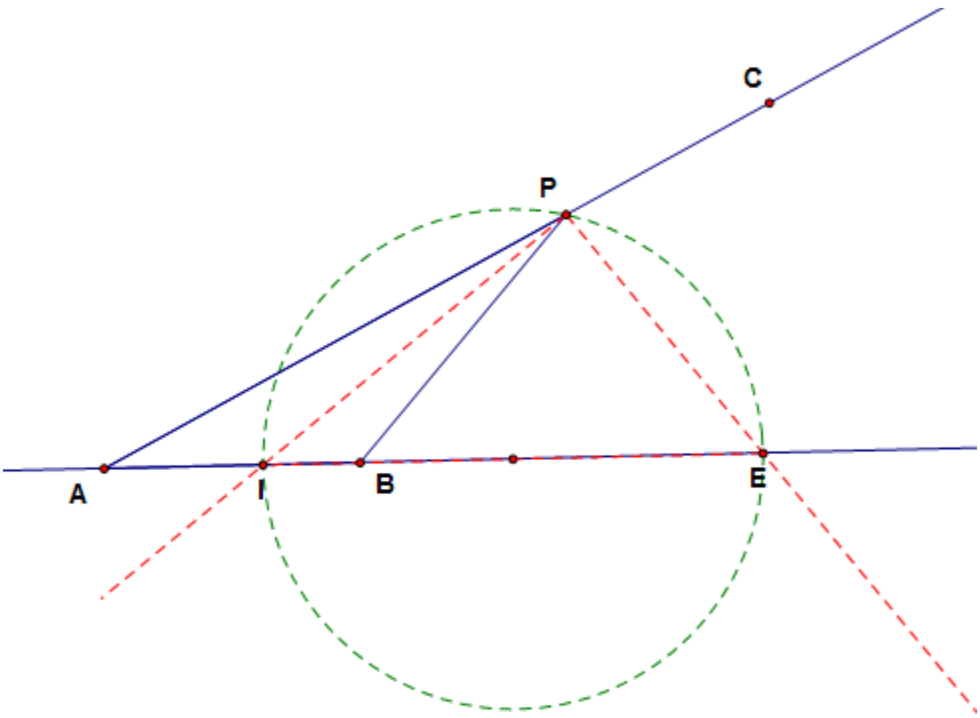
(i) The angle bisectors of a triangle

(ii) The proportion resulting from a line drawn parallel to a side of a triangle.

In fact, the proof connects these theorems in a nice way, and provides are refreshing idea about circles. The students can use Circle of Apollonius to tackle interesting, sometimes challenging, problems.

Let A and B be two fixed points, and let P be a point where PA: PB = k, a positive constant not equal to one. Next, bisect the angle APB internally and externally by PI and PE respectively, where I and

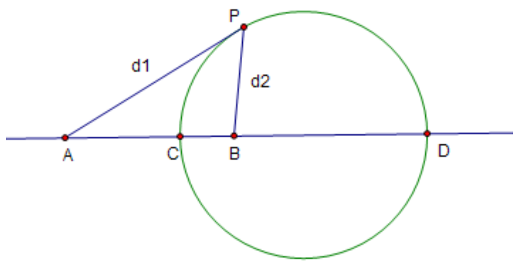
E are on AB.



Since PI bisects <APB and PE bisects its supplementary angle BPC, then by the bisector angle theorem, I and E divide AB internally and externally in the ratio k. Hence I and E are fixed points. Since m<IPE = 90°, the locus of P is a circle with IE being a diameter,

Conclusion: Apollonius's Alternative Definition of a circle

A circle is usually defined as the set of points P at a given distance r (the circle's radius) from a given point (the circle's center). However, there are other, equivalent definitions of a circle. Apollonius discovered that a circle could also be defined as the set of points P that have a given ratio of distances k = d1/d2 to two given points (labeled A and B in below Figure). These two points are sometimes called the foci.



Apollonian Circles Theorem[1]

The locus is a circle, unless of course r = 1, in which case it's the perpendicular bisector of AB. The proof exploits the properties of angle bisectors: internal and external. Construct points C and D on the line AB such that AC/BC = AD/BD= r. For r $\neq$ 1, C and D always exist. Note that both C and D lie on the sought locus.

The circle at hand has CD as a diameter. Indeed, C serves as the feet of the internal bisector of triangle APB at apex P, PD serves as the external bisector. Therefore, PC $\perp$PD. Chords PC and PD are perpendicular and therefore define a 90° inscribed angle. The angle subtends a 180° arc, which means that CD is a diameter of the circle.

For any P on the circle, the internal and external bisectors of angle APB pass through (the fixed points) C and D. Since, for the same A and B, each of the Apollonian circles corresponds to a different r, no two Apollonian circles intersect. For 0 < r < 1, the circles are closer to A and surround it. For smaller values of r, they are increasingly close to A. For r > 1, the circles surround B and, as r grows become

1 Retrieved on http://www.cut-the-knot.org/Curriculum/Geometry/LocusCircle.shtml#solution

increasingly close to it. For this reason, points A and B are often considered as circles, now *point circles*, circles of radius 0.

http://www.10ticks.co.uk/ies/vector/applets/apolon/apolon.html

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