

Functions Tangent To Their Product

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Problem:

Find two linear functions f and g such that f and g are each tangent to their product $h = f \cdot g$ at two distinct points.

Solution:

Let $f(x) = ax + b$ and $g(x) = cx + d$. We know that $a \neq 0$ and $c \neq 0$, otherwise the product is linear (vice quadratic), and the only way lines are tangent is if they coincide. So f and g each cross the x -axis at a unique point. If those two points are the same (say at $x = x_0$) then, since h only vanishes where either f or g vanish, h is tangent to the x -axis at x_0 . Then $h(x_0) = f(x_0) = g(x_0) = 0$ but $h'(x_0) = 0$ while $f'(x_0) \neq 0 \neq g'(x_0)$. Since h' is linear and thus monotonic, h can't have it's derivative switch sign enough to attain points of tangency with f and g .

So f and g must cross the x -axis at distinct points, say at $x = x_0$ and $x = x_1$ respectively. So $h(x_0) = h(x_1) = 0$ and h also crosses the x -axis at precisely those two points. The slope of h (the value of h') attains the slope of f at precisely one point (again by monotonicity and continuity of h'). If this does not happen at x_0 , then where it does happen, say at $x = x_2$, we would have $h(x_2) \neq f(x_2)$ since $h(x_0) = f(x_0) = 0$, again using monotonicity of h' . Therefore f and h must be tangent at x_0 . By symmetry, h must be tangent to g at x_1 .

So to begin let's fix two points on the x -axis to be the points of tangency. Let's make

the two points of tangency $(0, 0)$ and $(1, 0)$ for f and g respectively. Thus $0 = f(0) = b$ and $0 = g(1) = c + d$, so that $d = -c$. So our two linear functions are

$$\begin{aligned}f(x) &= ax \\g(x) &= cx - c\end{aligned}$$

So $h(x) = ax(cx - c) = acx(x - 1)$ and we see that h passes through our two points of tangency. To make the functions tangent we will impose conditions on the derivatives, so let's compute the derivatives.

$$\begin{aligned}f'(x) &= a \\g'(x) &= c \\h'(x) &= ac(x - 1 + x) = ac(2x - 1)\end{aligned}$$

The fact that f is tangent to h at the origin says that

$$f'(0) = h'(0) \Rightarrow a = -ac \Rightarrow a = 0 \text{ OR } c = -1$$

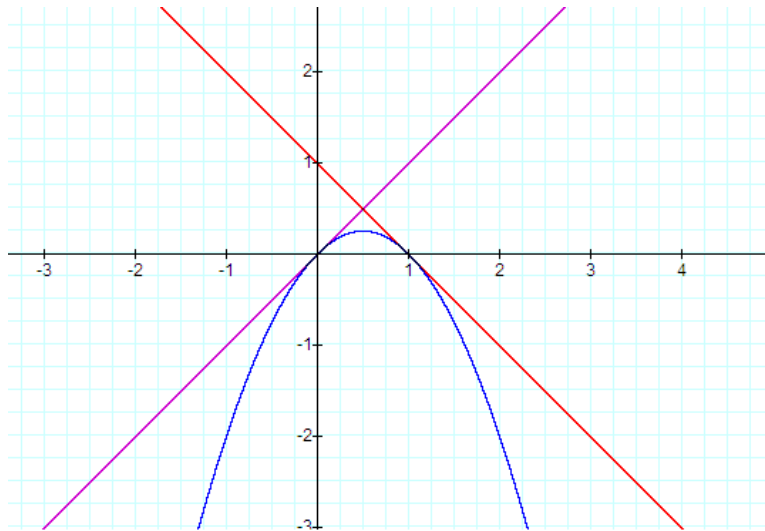
We know $a \neq 0$, since otherwise f and h are both identically 0 and the only way g could be tangent to h would be for g to be identically 0 and we would have at best a degenerate solution. So $c = -1$, and $g(x) = -x + 1$.

Imposing the tangency condition on g and h gives us

$$-1 = g'(1) = h'(1) = -a \Rightarrow a = 1$$

Let's check our solution. We have $f(x) = x$ and $g(x) = -x + 1$ so that

$h(x) = x(-x + 1) = -x^2 + x$. $f(0) = 0 = h(0)$ and $f'(0) = 1 = -2(0) + 1 = h'(0)$. Also we see that $g(1) = 0 = h(1)$ and $g'(1) = -1 = -2(1) + 1 = h'(1)$. So our solution works. We graph the solution below.



To illustrate the fact that the points of tangency must be on the x -axis, let's attempt to find a solution with tangencies at $(0, 0)$ and $(1, 1)$. Again we let $f(x) = ax + b$ and $g(x) = cx + d$. $f(0) = 0 \Rightarrow b = 0$. $g(1) = 1 \Rightarrow 1 = c + d$. Thus

$$f(x) = ax$$

$$g(x) = cx - c + 1$$

$$h(x) = acx^2 - acx + ax$$

Considering the derivatives we have $f'(0) = a$ and $h'(0) = -ac + a$. Therefore $a = -ac + a \Rightarrow 0 = -ac$ and either $a = 0$ or $c = 0$. Thus our product function is not quadratic and we have at best a degenerate solution. □

Problem:

Repeat the above problem, but now have $f(x)$ and $g(x)$ be quadratic functions such that each function, $f(x)$ and $g(x)$ is tangent to $h(x)$ in two different points. That is, $h(x)$ is a fourth degree equation and each of the second degree equations, $f(x)$ and $g(x)$ is tangent to $h(x)$ in two points.

Solution:

Again let's fix some reasonable points of tangency along the x -axis. Let's have them occur at x -values of 0, 1, 2, and 3. Considering how the graph of such a solution might look, let's have the points of tangency to f at 0 and 3 and g at 1, and 2 (see our final graphed solution).

Let $f(x) = a_2x^2 + a_1x + a_0$. $f(0) = 0 \Rightarrow a_0 = 0$. Also $f(3) = 0 \Rightarrow 0 = 9a_2 + 3a_1 \Rightarrow a_1 = -3a_2$. Thus $f(x) = a_2x^2 - 3a_2x$. Since we are down to one a let's set $a = a_2$ so that f is

$$f(x) = ax^2 - 3ax$$

Let $g(x) = c_2x^2 + c_1x + c_0$. $g(1) = 0 \Rightarrow 0 = c_2 + c_1 + c_0 \Rightarrow c_0 = -c_2 - c_1$. Also, $g(2) = 0 \Rightarrow 0 = 4c_2 + 2c_1 + c_0 = 4c_2 + 2c_1 - c_2 - c_1 = 3c_2 + c_1$. So that $c_1 = -3c_2$. Therefore g is

$$\begin{aligned} g(x) &= c_2x^2 - 3c_2x - c_2 - c_1 \\ &= c_2x^2 - 3c_2x - c_2 + 3c_2 \\ &= c_2x^2 - 3c_2x + 2c_2 \\ &= cx^2 - 3cx + 2c \end{aligned}$$

by taking $c = c_2$.

Now let's see if we can make the derivatives work out. First let's match the slope of f and h at 0.

$$f(x) = ax^2 - 3ax$$

$$f'(x) = 2ax - 3a$$

$$f'(0) = -3a$$

$$h(x) = (ax^2 - 3ax)(cx^2 - 3cx + 2c)$$

$$h'(x) = (ax^2 - 3ax)(2cx - 3c) + (2ax - 3a)(cx^2 - 3cx + 2c)$$

$$h'(0) = -3a(2c)$$

$$= -6ac$$

$$-3a = -6ac$$

$$c = \frac{1}{2}$$

Therefore $g(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$. Now let's match the slope of g and h at 1.

$$\begin{aligned}
g(x) &= \frac{1}{2}x^2 - \frac{3}{2}x + 1 \\
g'(x) &= x - \frac{3}{2} \\
g'(1) &= 1 - \frac{3}{2} \\
&= -\frac{1}{2} \\
h'(x) &= (ax^2 - 3ax)(2cx - 3c) + (2ax - 3a)(cx^2 - 3cx + 2c) \\
h'(1) &= (a - 3a)(2c - 3c) + (2a - 3a)(c - 3c + 2c) \\
&= (-2a)(-c) \\
&= 2ac \\
&= 2a \cdot \frac{1}{2} \\
&= a \\
a &= -\frac{1}{2}
\end{aligned}$$

Therefore $f(x) = -\frac{1}{2}x^2 + \frac{3}{2}x$.

Now let's check that this solution works. First observe that we setup f to have roots 0 and 3, and g to have roots 1 and 2. Since $h = f \cdot g$, h has those 4 roots and thus passes through our proposed points of tangency. So we just need to check the slopes at our proposed tangency points.

$$f(x) = -\frac{1}{2}x^2 + \frac{3}{2}x$$

$$f'(x) = -x + \frac{3}{2}$$

$$f'(0) = \frac{3}{2}$$

$$f'(3) = -3 + \frac{3}{2} = -\frac{3}{2}$$

$$g(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$$

$$g'(x) = x - \frac{3}{2}$$

$$g'(1) = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$g'(2) = 2 - \frac{3}{2} = \frac{1}{2}$$

$$h(x) = \left(-\frac{1}{2}x^2 + \frac{3}{2}x\right) \left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)$$

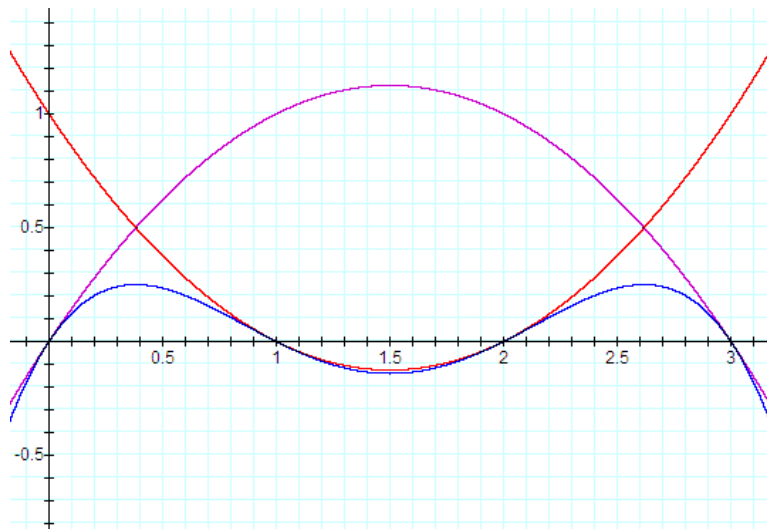
$$h'(x) = \left(-\frac{1}{2}x^2 + \frac{3}{2}x\right) \left(x - \frac{3}{2}\right) + \left(-x + \frac{3}{2}\right) \left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)$$

$$= \left(-x + \frac{3}{2}\right) \left[\frac{1}{2}x^2 - \frac{3}{2}x + \frac{1}{2}x^2 - \frac{3}{2}x + 1\right]$$

$$= \left(-x + \frac{3}{2}\right) (x^2 - 3x + 1)$$

$$\begin{aligned}
 h'(0) &= \frac{3}{2} \cdot 1 = \frac{3}{2} \\
 h'(1) &= \left(-1 + \frac{3}{2}\right)(1 - 3 + 1) \\
 &= \frac{1}{2}(-1) = -\frac{1}{2} \\
 h'(2) &= \left(-2 + \frac{3}{2}\right)(4 - 6 + 1) \\
 &= -\frac{1}{2}(-1) = \frac{1}{2} \\
 h'(3) &= \left(-3 + \frac{3}{2}\right)(9 - 9 + 1) \\
 &= -\frac{3}{2}(1) = -\frac{3}{2}
 \end{aligned}$$

We plot our solution below.



□

Comments:

One could attempt this problem without the use of calculus by experimenting with the graphs of the functions. One could setup the functions in graphing calculator using

parameters and then try different values of the parameters to make the graphs look right. Once one had a graph that looked right though, one would need some knowledge of calculus to actually prove that the graphs had the correct tangency requirements.