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Fibonacci and the Golden Mean

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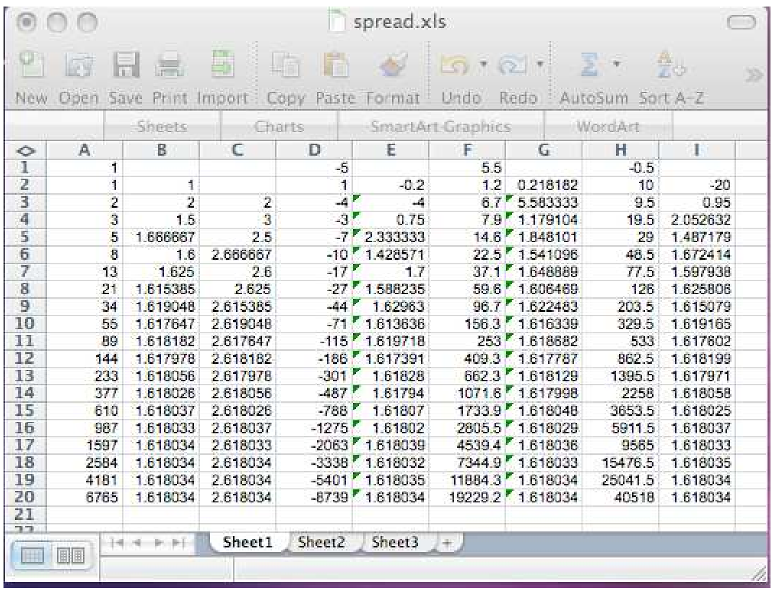

	A	B	C	D	E	F	G	H	I
1	1			-5		5.5		-0.5	
2	1	1		1	-0.2	1.2	0.218182	10	-20
3	2	2	2	-4	-4	6.7	5.583333	9.5	0.95
4	3	1.5	3	-3	0.75	7.9	1.179104	19.5	2.052632
5	5	1.666667	2.5	-7	2.333333	14.6	1.848101	29	1.487179
6	8	1.6	2.666667	-10	1.428571	22.5	1.541096	48.5	1.672414
7	13	1.625	2.6	-17	1.7	37.1	1.648889	77.5	1.597938
8	21	1.615385	2.625	-27	1.588235	59.6	1.606469	126	1.625806
9	34	1.619048	2.615385	-44	1.62963	96.7	1.622483	203.5	1.615079
10	55	1.617647	2.619048	-71	1.613636	156.3	1.616339	329.5	1.619165
11	89	1.618182	2.617647	-115	1.619718	253	1.618682	533	1.617602
12	144	1.617978	2.618182	-186	1.617391	409.3	1.617787	862.5	1.618199
13	233	1.618056	2.617978	-301	1.61828	662.3	1.618129	1395.5	1.617971
14	377	1.618026	2.618056	-487	1.61794	1071.6	1.617998	2258	1.618058
15	610	1.618037	2.618026	-788	1.61807	1733.9	1.618048	3653.5	1.618025
16	987	1.618033	2.618037	-1275	1.61802	2805.5	1.618029	5911.5	1.618037
17	1597	1.618034	2.618033	-2063	1.618039	4539.4	1.618036	9565	1.618033
18	2584	1.618034	2.618034	-3338	1.618032	7344.9	1.618033	15476.5	1.618035
19	4181	1.618034	2.618034	-5401	1.618035	11884.3	1.618034	25041.5	1.618034
20	6765	1.618034	2.618034	-8739	1.618034	19229.2	1.618034	40518	1.618034
21									

In the above spreadsheet shows, the first column is the Fibonacci sequence, $F_{i+1} = F_i + F_{i-1}$, with $F_0 = F_1 = 1$. The second column shows the ratios of successive terms. Anyone familiar with the golden mean, might notice that it appears that the second column is approaching the golden mean, $\frac{1+\sqrt{5}}{2} \approx 1.618$. The third column shows the ratio of every other term. If we suspect that the second column is approaching the golden mean then we should expect the third column to have the following limit

$$\begin{aligned}\lim_{i\to\infty}\frac{F_{i+2}}{F_i} &= \lim_{i\to\infty}\frac{F_{i+2}}{F_{i+1}}\cdot\frac{F_{i+1}}{F_i}\\ &= \lim_{i\to\infty}\frac{F_{i+2}}{F_{i+1}}\cdot\lim_{i\to\infty}\frac{F_{i+1}}{F_i}\\ &= \left(\frac{1+\sqrt{5}}{2}\right)^2\\ &\approx 2.618\end{aligned}$$

and it appears that this is the case. The fourth, sixth, and eighth columns are examples of Lucas sequences, which are sequences with the same recursion relation as the Fibonacci sequence; they just start with values other than two ones. The columns immediately to the right of each Lucas sequence shows the ratios of successive terms. It appears that each of these limits is also the golden mean. We will in fact prove this.

Problem: Prove that the ratio of successive terms of the Fibonacci sequence and all Lucas sequences (not seeded with two zeros) converge to the golden mean, $\frac{1}{2}(1+\sqrt{5})$.

Solution: We denote the Fibonacci sequence by F_n, with $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n > 0$. We will use the method of (simple) continued fractions to show that $\lim_{n\to\infty}\frac{Fn+1}{F_n} = \frac{1}{2}(1+\sqrt{5})$.

An expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

with $a_n \in \mathbb{Z}$ for all n, and $a_n > 0$ for $n \geq 1$ is said to be a (simple) continued fraction. (Simple means that we have a 1 in all the numerators. Since we will only be concerned with simple continued fractions, we will drop the word "simple" from here on out.) For notational convenience we denote the above expression by

$$[a_0; a_1, a_2, ...] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + ...}}$$

Certainly, if our sequence is finite, i.e. if we are considering $[a_0; a_1, a_2, ..., a_n]$, then since all the a_i for $i \geq 1$ are positive, we have no chance of dividing by zero and $[a_0; a_1, a_2, ..., a_n] \in \mathbb{Q}$.

In the finite case, to actually compute $[a_0; a_1, a_2, ..., a_n]$, the straightforward way to proceed would be to start on the right, with a_n, and work from right to left. Our first goal is to make sense of an infinite continued fraction, and in that case working from right to left won't be possible. Thus our first Lemma gives us a way to compute a continued fraction from left to right.

Lemma: Let $[a_0; a_1, a_2, ..., a_n]$ be a continued fraction. Define two sequences by $p_{-2} = 0$, $p_{-1} = 1$ and for all $i \geq 0$, $p_i = a_i p_{i-1} + p_{i-2}$, and $q_{-2} = 1$, $q_{-1} = 0$ and for all $i \geq 0$, $q_i = a_i q_{i-1} + q_{i-2}$. Then $[a_0; a_1, a_2, ..., a_n] = \frac{p_n}{q_n}$.

Proof. We will induct on the length of the continued fraction. For $n = 0$, we have

$$\begin{aligned} [a_0] &= a_0 \\ &= \frac{a_0 \cdot 1 + 0}{a_0 \cdot 0 + 1} \\ &= \frac{a_0 p_{-1} + p_{-2}}{a_0 q_{-1} + q_{-2}} \\ &= \frac{p_0}{q_0} \end{aligned}$$

Now suppose that for some $n \geq 0$ we have $[a_0; a_1, a_2, ..., a_n] = \frac{p_n}{q_n}$. Then

$$\begin{aligned}
[a_0; a_1, a_2, ..., a_n, a_{n+1}] &= [a_0; a_1, a_2, ..., a_n + 1/a_{n+1}] \\
&= \frac{(a_n + \frac{1}{a_n+1})p_{n-1} + p_{n-2}}{(a_n + \frac{1}{a_n+1})q_{n-1} + q_{n-2}} \\
&= \frac{a_{n+1}a_n p_{n-1} + p_{n-1} + a_{n+1}p_{n-2}}{a_{n+1}a_n q_{n-1} + q_{n-1} + a_{n+1}q_{n-2}} \\
&= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}} \\
&= \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} \\
&= \frac{p_{n+1}}{q_{n+1}}
\end{aligned}$$

□

In the infinite case, we define $[a_0; a_1, a_2, ...] = \lim_{n\to\infty} [a_0; a_1, a_2, ..., a_n]$. Our next goal is to show that this limit always exists. The next Lemma will help us show this.

Lemma: With the sequences $\{a_i\}$, $\{p_i\}$, and $\{q_i\}$ as above, $p_n q_{n-1} - p_{n-1}q_n = (-1)^{n-1}$, for all $n \geq -1$.

Proof. Again, we proceed by induction. For $n = -1$ we have

$$p_{-1}q_{-2} - p_{-2}q_{-1} = 1 \cdot 1 - 0 \cdot 0 = 1 = (-1)^{-2}$$

Suppose this is true for some $n \geq -1$. Then

$$\begin{aligned}
p_{n+1}q_n - p_n q_{n+1} &= (a_{n+1}p_n + p_{n-1})q_n - (a_{n+1}q_n + q_{n-1})p_n \\
&= -(p_n q_{n-1} - p_{n-1}q_n) \\
&= -(-1)^{n-1} \\
&= (-1)^n
\end{aligned}$$

□

Thus

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}$$

Now $q_0 = 1$, $q_1 = a_1$, and $q_2 = a_2 a_1 + 1$. Thus $\{q_i\}_{i=1}^{\infty}$ is a strictly increasing sequence of positive integers and therefore so is $\{q_{i+1}q_i\}_{i=1}^{\infty}$. Therefore for n odd

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}} = \frac{1}{q_n q_{n-1}} > 0$$

and for n even

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}} = \frac{-1}{q_n q_{n-1}} < 0$$

Also

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} + \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^{n-1}}{q_n q_{n-1}} + \frac{(-1)^{n-1}}{q_{n-1}q_{n-2}}$$

So for n odd,

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{1}{q_n q_{n-1}} - \frac{1}{q_{n-1}q_{n-2}} < 0$$

and for n even

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{1}{q_n q_{n-1}} - \frac{1}{q_{n-1}q_{n-2}} > 0$$

Putting everything together we have

Theorem: In the sequence $\{\frac{p_i}{q_i}\}_{i=0}^{\infty}$, every even term is less than every odd term, the even terms form an increasing subsequence, and the odd terms form a decreasing subsequence. Thus the even subsequence has a limit as well as the odd subsequence. Since $\frac{1}{q_n q_{n-1}} \to 0$, both these limits coincide and thus we can define

$$[a_0; a_1, ...] = \lim_{i\to\infty}[a_0; a_1, ..., a_i] = \lim_{i\to\infty}\frac{p_i}{q_i}$$

Now we can apply this to the Fibonacci sequence by considering the infinite continued fraction $[1; 1, 1, 1, ...]$. The sequence of p's is $\{0, 1, 1, 2, 3, 5, ...\}$ while the q's are $\{1, 0, 1, 1, 2, 3, 5, ...\}$. So both these sequences are the Fibonacci sequence, just shifted by one, so

$$[1; 1, 1, 1, ...] = \lim_{i\to\infty}\frac{p_i}{q_i} = \lim_{i\to\infty}\frac{F_{i+1}}{F_i}$$

Letting x be the value of this limit we have

$$\begin{aligned} x &= 1 + \frac{1}{1 + \frac{1}{1+...}} \\ &= 1 + \frac{1}{x} \\ x^2 &= x + 1 \\ x^2 - x - 1 &= 0 \\ x &= \frac{1}{2}(1 \pm \sqrt{5}) \end{aligned}$$

But since $x > 1$, $x = \frac{1}{2}(1+\sqrt{5})$.

Now we consider the Lucas sequence starting at a and b. Thus our sequence is $\{a, b, a+b, a+2b, 2a+3b, 3a+5b, ..., F_i a + F_{i+1} b, ...\}$. If $a = b = 0$, then our sequence is identically zero and we can't consider ratios of successive terms. If one of a and b is zero, then the ratios are precisely the ratios of the Fibonacci sequence, so we consider the nontrivial case where both $a \neq 0$ and $b \neq 0$. Scaling the sequence by a nonzero constant, will not affect the ratios of successive terms, so we can scale the sequence by $1/a$, or assume $a = 1$.

We consider two cases.

Case 1: $b \neq -\frac{2}{1+\sqrt{5}}$. Thus

$$\begin{aligned}\lim_{i\to\infty} \frac{F_i + F_{i+1}b}{F_{i-1} + F_i b} &= \lim_{i\to\infty} \frac{1 + \frac{F_{i+1}}{F_i}b}{\frac{F_{i-1}}{F_i} + b} \\ &= \frac{1 + \frac{1+\sqrt{5}}{2}b}{\frac{2}{1+\sqrt{5}} + b}\end{aligned}$$

If we let

$$g(b) = \frac{1 + \frac{1+\sqrt{5}}{2}b}{\frac{2}{1+\sqrt{5}} + b}$$

then g is a differentiable for all $b \neq -\frac{2}{1+\sqrt{5}}$.

Our aim is to show that g is constant (and equal to the golden mean).

So

$$
\begin{aligned}
g'(b) &= \frac{\left(\frac{2}{1+\sqrt{5}}+b\right)\left(\frac{1+\sqrt{5}}{2}\right)-\left(1+\frac{1+\sqrt{5}}{2}b\right)}{\left(\frac{2}{1+\sqrt{5}}+b\right)^2} \\
&= \frac{1+\frac{1+\sqrt{5}}{2}b-1-\frac{1+\sqrt{5}}{2}b}{\left(\frac{2}{1+\sqrt{5}}+b\right)^2} \\
&= 0
\end{aligned}
$$

So g is locally constant, and since $\lim_{b\to\pm\infty} g(b) = \frac{1+\sqrt{5}}{2}$, $g(b) = \frac{1+\sqrt{5}}{2}$ for all $b \neq -\frac{2}{1+\sqrt{5}}$.

Case 2: $b = -\frac{2}{1+\sqrt{5}}$.

We compute as follows

$$
\begin{aligned}
\lim_{i\to\infty} \frac{F_i + F_{i+1}b}{F_{i-1}+F_i b} &= \lim_{i\to\infty} \frac{1+\frac{F_{i+1}}{F_i}b}{\frac{F_{i-1}}{F_i}+b} \\
&= \lim_{x\to\frac{1+\sqrt{5}}{2}} \frac{1+x\left(-\frac{2}{1+\sqrt{5}}\right)}{\frac{1}{x}-\frac{2}{1+\sqrt{5}}} \\
&= \lim_{x\to\frac{1+\sqrt{5}}{2}} \frac{-\frac{2}{1+\sqrt{5}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\frac{1+\sqrt{5}}{2}} x^2 \frac{2}{1+\sqrt{5}} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^2 \frac{2}{1+\sqrt{5}} \\
&= \frac{1+\sqrt{5}}{2}
\end{aligned}
$$

Thus we are done.