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The xa, xb, and xc Planes for $ax^2 + bx + c = 0$

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We are going to first look at some graphs of the equation $ax^2 + bx + c = 0$ in the xb plane. First let's set $a = 1$. Thus we are considering $x^2 + bx + c = 0$, which has solutions at $x = \frac{-b \pm \sqrt{b^2-4c}}{2}$. If $b^2 - 4c > 0$ then we have two distinct real roots. We are considering x and b as the variables and c as a constant, so we see that we will have two distinct real roots for all b if $c < 0$. We demonstrate this in the next graph, by taking $c = -1, -2, -10$ and plotting these all in the same xb-plane.

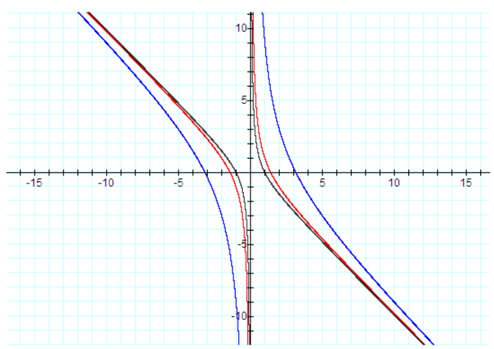

Figure 1: graphs of $x^2 + bx + c = 0$ in an xb-plane with $c = -1, -2, -10$

The black graph is $c = -1$, red is $c = -2$, and blue is $c = -10$. We see that any horizontal line crosses each graph exactly twice. Which means that for $c < 0$ and for any b, $x^2 + bx + c = 0$ has two solutions.

Next we consider the case $c > 0$. Now our discriminant of $b^2 - 4c$ can be negative, positive or zero. If the discriminant is negative, we should not have any roots. If it is positive, we should have two real roots and if it is zero, we should have one real root. Figure 2 shows three plots with $c = 1, 2, 10$. The black graph is $c = 1$. With $c = 1$, we should have two real roots when $b^2 > 4$, or when $b > 2$ or $b < -2$. When $b = \pm 2$, we should only have one solution, and when $-2 < b < 2$, no solutions. The graph confirms this (look at horizontal lines intersecting the graph at appropriate b-values. On the red graph, $c = 2$ and we see the phenomena of changing the number of roots change at $b^2 - 4(2) = 0$, or at $b = \pm\sqrt{8} \approx 2.8$. With $c = 10$ (the blue graph), the change occurs at $b^2 = 40$ or $b = \pm\sqrt{40} \approx 6.3$.

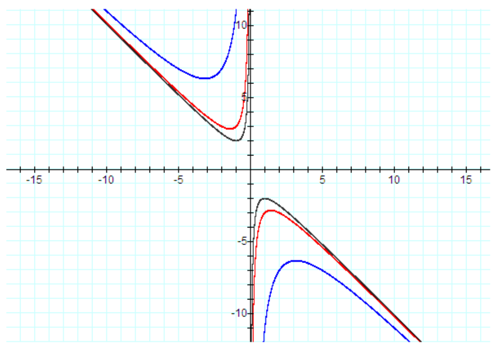

Figure 2: graphs of $x^2 + bx + c = 0$ in an xb-plane with $c = 1, 2, 10$

So what happens in between these two regimes, when $c = 0$. Well, then we have the equation $x^2 + bx = 0$. Which factors as $x(x + b) = 0$. So we get all the points where either $x = 0$ or where $b = -x$. Thus we get a line with slope -1 and the b-axis. We show the graph below.

You might say that the b-axis isn't there. I still claim it is, it's just that graphing calculator decided to cover it up. To see that b-axis is part of the graph, we can shift everything to the right 1 unit, by plotting $(x - 1)^2 + b(x - 1) = 0$.

So far the entire discussion has been with $a = 1$. So let's quickly see what changes if $a < 0$, say $a = -1$. With $a = -1$ the discriminant is $b^2 + 4c$, so we should always have two real roots for $c > 0$ and places with 0, 1, and 2 roots if $c < 0$. The following graph shows this, with $c = 1$ in red and $c = -1$ in blue.

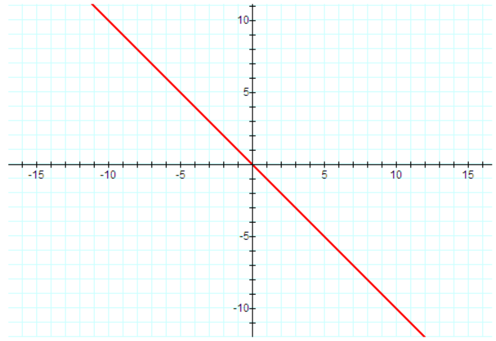

Figure 3: graph of $x^2 + bx + c = 0$ in an xb-plane with $c = 0$

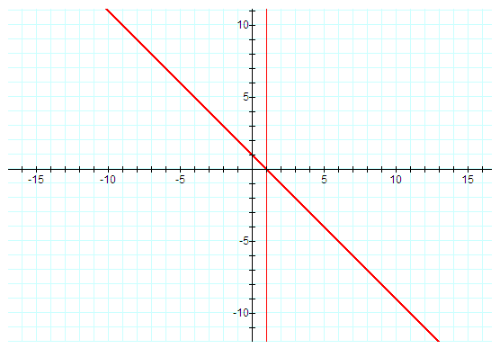

Figure 4: graph of $(x-1)^2 + b(x-1) = 0$ in an xb-plane

Considering the quadratic equation,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

when a, b, and c are fixed (and the discriminant is positive) we have the point $x = \frac{-b}{2a}$, lying at the average of the two solutions. $x = \frac{-b}{2a} \Rightarrow 2ax + b = 0$. Thus if we allow b to vary with x, we expect the line $2ax + b = 0$ to "cut the graph down the middle". Note this line does not depend on c. So we plot $x^2 + bx + 1 = 0$, $x^2 + bx - 1 = 0$, and $2x + b = 0$ on the same graph and see that we get the expected result (that the line stays right in the middle of the two graphs when viewed along horizontal cross sections.) To illustrate the same phenomena when $a < 0$, we include a graph of $-x^2 + bx + 1 = 0$, $-x^2 + bx - 1 = 0$, and $-2x + b = 0$.

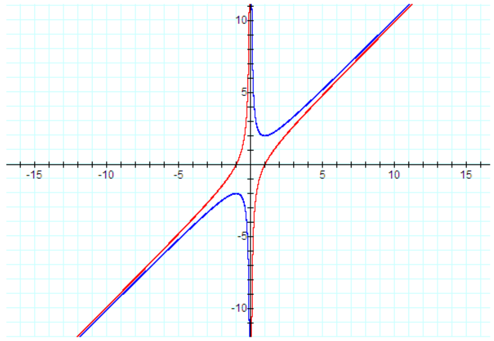

Figure 5: graph of $-x^2 + bx + c = 0$ in an xb-plane with $c = \pm 1$

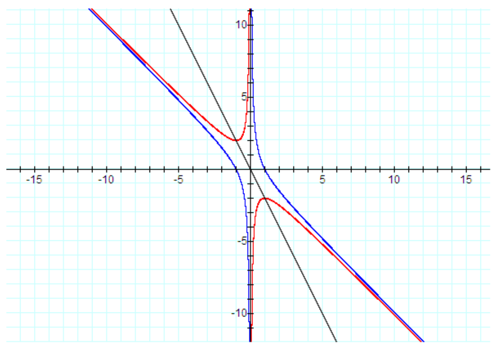

Figure 6: graph of $x^2 + bx + 1 = 0$, $x^2 + bx - 1 = 0$, and $2x + b = 0$ in an xb-plane

Let's move on to the other possible planes. The easiest being the xc-plane. With a and b fixed we can solve for c as $c = -ax^2 - bx$ which is a parabola which faces upwards for $a < 0$ and downwards for $a > 0$. If $a = 0$ we get $c = -bx$, which is a line with slope $-b$ in the xc-plane (which holds even for $b = 0$). If $a \neq 0$, we have $c = -x(ax + b)$, which has two roots, one at $x = 0$ and one at $x = -\frac{b}{a}$. So if $b = 0$, we have the parabola that is tangent to the x-axis. Otherwise there are two distinct roots. So far $a \neq 0$ we always have parabolas and thus horizontal lines will cross 0, 1, or 2 times.

Finally let's explore the xa-plane. Let's start with $b = c = 1$. So we are considering $ax^2 + x + 1 = 0$ and solving for a gives $a = \frac{-x-1}{x^2}$. Thus a is a rational function of x that limits to zero as $x \to \pm\infty$. Also as $x \to 0$ from either side the numerator approaches -1 and the denominator goes to zero through positive numbers. Thus we should have a vertical

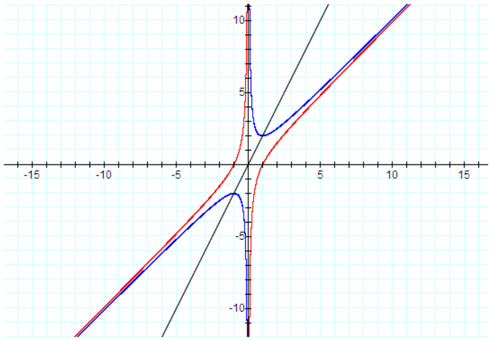

Figure 7: graph of $-x^2 + bx + 1 = 0$, $-x^2 + bx - 1 = 0$, and $-2x + b = 0$ in an xb-plane

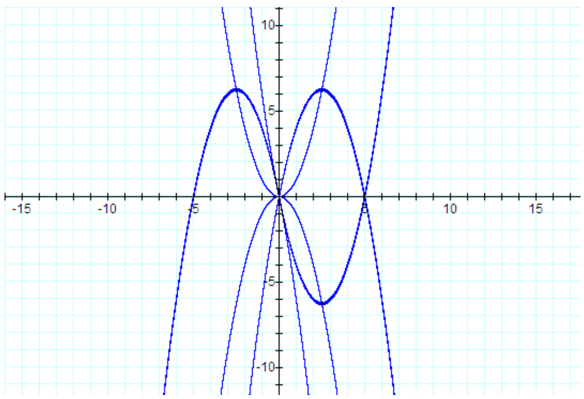

Figure 8: graphs of $ax^2 + bx + c = 0$ in an xc-plane

asymptote at $x = 0$ that goes to $-\infty$ on both sides. Furthermore, we should only have one zero, at $x = -1$. Let's consider da/dx.

$$
\begin{aligned}
\frac{da}{dx} &= \frac{(x^2)(-1) - (-x-1)(2x)}{x^4} \\
&= \frac{-x^2 + 2x^2 + 2x}{x^4} \\
&= \frac{x^2 + 2x}{x^4} \\
&= \frac{x+2}{x^3}
\end{aligned}
$$

Thus the sign of da/dx is the same as $x(x+2)$, which increases on $(-\infty, -2)$ and $(0, \infty)$, and decreases on $(-2, 0)$. So a has a maximum value of $a = \frac{1}{4}$ at $x = -2$. Thus for $a < 0$ we have two roots. For $a = 0$, we have one root. For $a \in (0, 1/4)$, we have two roots. For $a = 1/4$, there is one root and for $a > 1/4$, zero roots. As before, where there are two roots, $x = -b/2a$ is at their arithmetic mean. Thus the hyperbola $ax = -b/2 = -1/2$, runs "right down the middle" where there are two roots. We include the graph below.

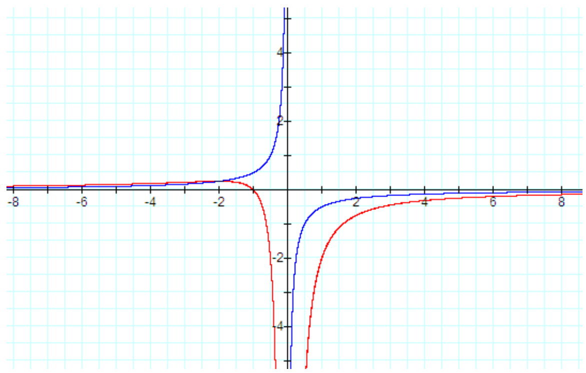

Figure 9: graph of $ax^2 + x + 1 = 0$ in red and $ax = -1/2$ in blue in an xa-plane

If we change $c = -1$, then we have the rational function $a = \frac{-x+1}{x^2}$. This function still limits to zero at infinity, but now the only zero occurs at $x = 1$. Also the vertical asymptote goes to $+\infty$ from both sides. In this case $da/dx = \frac{x-2}{x^3}$, so that a increases on $(-\infty, 0)$ and $(2, \infty)$, and decreases on $(0, 2)$. So for $a > 0$ and $a \in (-1/4, 0)$ there are two roots. There is one root for $a = 0, -1/4$, and no roots for $a < -1/4$.

Finally let's consider what happens when either $b = 0$ or $c = 0$. For $b = 0$ and $c = 1$, we have $a = -\frac{1}{x^2}$, whose graph we know and show in red in the final figure. Thus for $a < 0$, there are two roots, and for $a \geq 0$, no roots. For $b = 1$ and $c = 0$, we have $ax^2 + x = x(ax+1) = 0$. Thus for each a, there is always two roots, one at $x = 0$ and one at $x = -1/a$. Thus we get the picture in the xa-plane in blue (plus the a-axis).

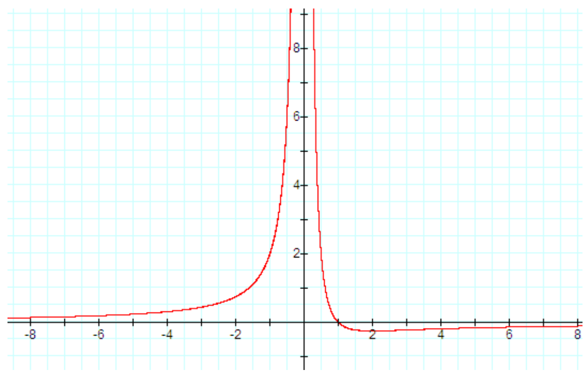

Figure 10: graph of $ax^2 + x - 1 = 0$ in an xa-plane

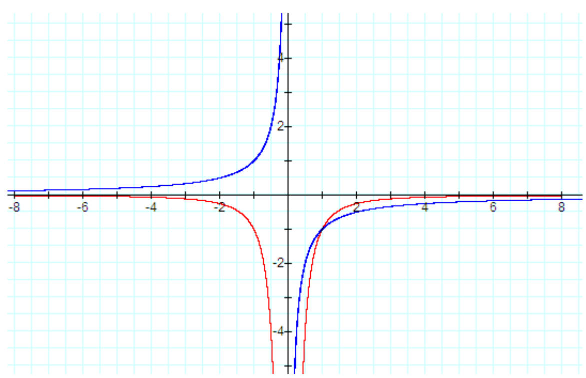

Figure 11: graph of $ax^2 + 1 = 0$ in red and $ax^2 + x = 0$ in blue in an xa-plane