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The Centroid, Orthocenter, and Circumcenter of a Triangle

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Problem: Prove that for any triangle, H (the orthocenter), G (the centroid), and C (the circumcenter) are collinear, and prove that $|HG| = 2|GC|$.

Solution: Given a triangle in the plane, we can choose coordinates on the plane such that one vertex is at $(0,0)$, the second is at $(1,0)$, and the third is at (a,b). We first show that the centroid is well-defined and work out its coordinates.

Case 1: $a \neq -1, 2, \frac{1}{2}$

So the three midpoints of the sides of the triangle are located at coordinates $(1/2, 0)$, $(a/2, b/2)$, and $((a+1)/2, b/2)$. Let $L1$ be the median connecting $(0,0)$ to $((a+1)/2, b/2)$, $L2$ be the median connecting $(1,0)$ to $(a/2, b/2)$, and $L3$ be the median connecting (a,b) to $(1/2, 0)$. So $L1$ is given by

$$L1: \quad y = \frac{b/2}{(a+1)/2}x = \frac{b}{a+1}x$$

$L2$ is given by

$$L2: \quad y = \frac{b/2}{a/2-1}(x-1) = \frac{b}{a-2}(x-1)$$

and $L3$ is given by

$$L3: \quad y = \frac{b}{a-1/2}(x-1/2) = \frac{b}{2a-1}(2x-1)$$

Let's solve for the intersection point of $L1$ and $L2$. In the computation we will use that $b \neq 0$ otherwise the 3 vertices would be colinear and we would not have a triangle. Thus our first step is to divide by b.

$$\begin{aligned} \frac{b}{a+1}x &= \frac{b}{a-2}(x-1) \\ \frac{x}{a+1} &= \frac{x-1}{a-2} \\ ax-2x &= ax-a+x-1 \\ a+1 &= 3x \\ x &= \frac{a+1}{3} \end{aligned}$$

Substituting into the equation for $L1$ (and $L2$) gives us that $y = b/3$. So $L1$ intersects $L2$ at $\left(\frac{a+1}{3}, \frac{b}{3}\right)$. Now we just need to check that this point is on $L3$.

$$\frac{b}{2a-1}\left(2 \cdot \frac{a+1}{3} - 1\right) = \frac{b}{3} \cdot \frac{1}{2a-1}(2(a+1)-3) = \frac{b}{3} \cdot \frac{2a-1}{2a-1} = \frac{b}{3}$$

So all three medians intersect in a common point.

Case 2: $a = 1/2$.

In this case, our three midpoints are located at $(1/2, 0)$, $(1/4, b/2)$, and $(3/4, b/2)$.

Our three medians thus have equations

$$L1: \quad y = \frac{b/2}{3/4}x = \frac{2b}{3}x$$

$$L2: \quad y = \frac{b/2}{-3/4}(x-1) = -\frac{2b}{3}(x-1)$$

$$L3: \quad x = \frac{1}{2}$$

So we find where $L1$ and $L2$ intersect.

$$\begin{aligned}
\frac{2b}{3}x &= -\frac{2b}{3}(x-1) \\
x &= -x+1 \\
2x &= 1 \\
x &= 1/2 \\
y &= \frac{2b}{3}\cdot\frac{1}{2} = \frac{b}{3}
\end{aligned}$$

Since the point $(1/2, b/3)$ is also on $L3$ the three medians intersect in a common point.

Case 3: $a = -1$.

Our three midpoints are at $(0, b/2)$, $(-1/2, b/2)$, and $(1/2, 0)$. Our three medians have equations

$$L1: \quad x = 0$$

$$L2: \quad y = \frac{b/2}{-1/2-1}(x-1) = -\frac{b/2}{-3/2}(x-1) = -\frac{b}{3}(x-1)$$

$$L3: \quad y = \frac{b}{-1-1/2}(x-1/2) = \frac{b}{-3/2}(x-1/2) = -\frac{b}{3}(2x-1)$$

Now we compute the intersection of $L2$ and $L3$.

$$
\begin{aligned}
-\frac{b}{3}(x-1) &= -\frac{b}{3}(2x-1) \\
x-1 &= 2x-1 \\
x &= 0 \\
y &= -\frac{b}{3}(-1) = \frac{b}{3}
\end{aligned}
$$

Thus $(0, b/3)$ is on all three medians.

Case 4: $a = 2$.

Our three midpoints are at $(1/2, 0)$, $(3/2, b/2)$, and $(1, b/2)$. The equations of our three medians is therefore

$$L1: \qquad y = \frac{b/2}{3/2}x = \frac{b}{3}x$$

$$L2: \qquad x = 1$$

$$L3: \qquad y = \frac{b}{2-1/2}(x-1/2) = \frac{b}{4-1}(2x-1) = \frac{b}{3}(2x-1)$$

Solving for the intersection of $L1$ with $L3$ gives

$$
\begin{aligned}
\frac{b}{3}x &= \frac{b}{3}(2x-1) \\
x &= 2x-1 \\
x &= 1 \\
y &= \frac{b}{3}(1) = \frac{b}{3}
\end{aligned}
$$

Therefore, this intersection point is on $L2$ also. Thus in all cases, the three medians intersect in a common point, the centroid, and it has coordinates $G = (\frac{a+1}{3}, \frac{b}{3})$.

Next, we show that the orthocenter, H is well-defined and work out its coordinates. We recall that H is defined to be the intersection of the three altitudes of the triangle. Let $S1$ be the side of the triangle with vertices $(0,0)$ and $(1,0)$. Let $L1$ be the altitude perpendicular to $S1$. Thus $L1$ has equation

$$L1 : x = a$$

Let $S2$ be the side of the triangle with vertices $(0,0)$ and (a,b), and let $L2$ be the altitude perpendicular to $S2$. Assume momentarily that $a \neq 0$. Then the slope of $S2$ is $\frac{b}{a}$. So $L2$, being perpendicular to $S2$ and passing through $(1,0)$ has equation

$$L2 : y = -\frac{a}{b}(x-1)$$

If $a = 0$, then $L2$ is given by $y = 0$, so our equation for $L2$ is valid in all cases.

Let $S3$ be the side of the triangle with vertices $(1,0)$ and (a,b), and let $L3$ be the altitude perpendicular to $S3$. Assume momentarily that $a \neq 1$. Then the slope of $S3$ is $\frac{b}{a-1}$. So $L3$ has equation

$$L3 : y = \frac{1-a}{b}x$$

If $a = 1$, then $L3$ is given by $y = 0$, so our equation for $L3$ is valid in all cases.

Now we solve for the intersection point of $L2$ and $L3$.

$$\begin{aligned} -\frac{a}{b}(x-1) &= \frac{1-a}{b}x \\ -ax+a &= x-ax \\ x &= a \\ y &= \frac{a(1-a)}{b} \end{aligned}$$

This point is on $L1$ also, so the orthocenter H is well-defined and has coordinates $H = \left(a, \frac{a(1-a)}{b}\right)$.

Now we show that the circumcenter, C, is well-defined and work out its coordinates. We will keep the sides of the triangle labeled as above. Let $L1$ be the perpendicular bisector to $S1$. Thus $L1$ is given by $x = 1/2$.

Let $L2$ be the perpendicular bisector to $S2$. So $L2$ has equation

$$L2 : y - \frac{b}{2} = -\frac{a}{b}\left(x - \frac{a}{2}\right)$$

Let $L3$ be the perpendicular bisector to $S3$. So $L3$ has equation

$$L3 : y - \frac{b}{2} = \frac{1-a}{b}\left(x - \frac{a+1}{2}\right)$$

So we compute $L2 \cap L3$ as

$$\begin{aligned}
-\frac{a}{b}\left(x-\frac{a}{2}\right) &= \frac{1-a}{b}\left(x-\frac{a+1}{2}\right) \\
-ax+\frac{a^2}{2} &= x-\frac{a+1}{2}-ax+\frac{a(a+1)}{2} \\
\frac{a^2}{2} &= x-\frac{a}{2}-\frac{1}{2}+\frac{a^2}{2}+\frac{a}{2} \\
x &= 1/2 \\
y &= -\frac{a}{b}\left(\frac{1}{2}-\frac{a}{2}\right)+\frac{b}{2} \\
&= \frac{a(a-1)}{2b}+\frac{b}{2}
\end{aligned}$$

Since this point is on $L1$ also, C is well-defined and has coordinates

$$C=\left(\frac{1}{2},\frac{a(a-1)}{2b}+\frac{b}{2}\right)$$

Now we show that G, H, and C are colinear.

Case 1: G and H sit on a common vertical line.

Then

$$\begin{aligned}
\frac{a+1}{3} &= a \\
a+1 &= 3a \\
1 &= 2a \\
a &= \frac{1}{2}
\end{aligned}$$

Thus C sits on the same vertical line $x=1/2$.

Case 2: G and H do not sit on a common vertical line. Therefore $a \neq 1/2$ and in the computations that follow, none of the denominators are 0. We will compare the slope of GH to the slope of HC (we denote the slope function as $m(\cdot)$). The slope of GH is

$$\begin{aligned} m(GH) &= \frac{\frac{b}{3} - \frac{a(1-a)}{b}}{\frac{a+1}{3} - a} \cdot \frac{3b}{3b} \\ &= \frac{b^2 - 3a(1-a)}{(a+1)b - 3ab} \\ &= \frac{b^2 - 3a + 3a^2}{ab + b - 3ab} \\ &= \frac{b^2 - 3a + 3a^2}{b - 2ab} \end{aligned}$$

The slope of HC is

$$\begin{aligned} m(HC) &= \frac{\frac{a(a-1)}{2b} + \frac{b}{2} - \frac{a(1-a)}{b}}{\frac{1}{2} - a} \cdot \frac{2b}{2b} \\ &= \frac{a^2 - a + b^2 - 2a(1-a)}{b - 2ab} \\ &= \frac{a^2 - a + b^2 - 2a + 2a^2)}{b - 2ab} \\ &= \frac{3a^2 - 3a + b^2}{b - 2ab} \end{aligned}$$

Thus $m(GH) = m(HC)$ and G, H, and C are colinear.

Now for our final computation. We have

$$\begin{aligned}
|HG|^2 &= \left(\frac{a+1}{3} - a\right)^2 + \left(\frac{a(1-a)}{b} - \frac{b}{3}\right)^2 \\
&= \frac{1}{9}(a+1-3a)^2 + \frac{1}{9b^2}(3a(1-a) - b^2)^2 \\
&= \frac{1}{9}(1-2a)^2 + \frac{1}{9b^2}(3a - 3a^2 - b^2)^2 \\
&= \frac{1}{9}(2a-1)^2 + \frac{1}{9b^2}(3a^2 - 3a + b^2)^2 \\
&= \frac{1}{9}(2a+2-3)^2 + \frac{1}{9b^2}(3a^2 - 3a + b^2)^2 \\
&= \frac{1}{9}(2(a+1)-3)^2 + \frac{1}{9b^2}(3a(a-1) + 3b^2 - 2b^2)^2 \\
&= 4\left[\left(\frac{a+1}{3} - \frac{1}{2}\right)^2 + \left(\frac{a(a-1)}{2b} + \frac{b}{2} - \frac{b}{3}\right)^2\right] \\
&= 4|GC|^2
\end{aligned}$$

So

$$|HG| = 2|GC|$$

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