Maximum Picture Viewing Angle

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Problem: A 4 by 4 picture hangs on a wall such that its bottom edge is 2 ft above your eye level. How far back from the picture should you stand, directly in front of the picture, in order to view the picture under the maximum angle?

Solution: Let the distance $x$ and angles $a$ and $b$ be as in the figure below.

Thus $\tan(a + b) = \frac{6}{x} \Rightarrow a + b = \tan^{-1} \frac{6}{x}$. Also we have $\tan b = \frac{2}{x} \Rightarrow b = \tan^{-1} \frac{2}{x}$. So

$$a = \tan^{-1} \frac{6}{x} - b = \tan^{-1} \frac{6}{x} - \tan^{-1} \frac{2}{x}$$
Therefore
\[
\frac{da}{dx} = \frac{1}{1 + \frac{36}{x^2}} \cdot \frac{-6}{x^2} - \frac{1}{1 + \frac{4}{x^2}} \cdot \frac{-2}{x^2}
\]
\[
= \frac{-6}{x^2 + 36} + \frac{2}{x^2 + 4}
\]

Setting \( da/dx = 0 \) gives us
\[
\frac{6}{x^2 + 36} = \frac{2}{x^2 + 4}
\]
\[
\frac{3}{x^2 + 36} = \frac{1}{x^2 + 4}
\]
\[
3x^2 + 12 = x^2 + 36
\]
\[
2x^2 = 24
\]
\[
x^2 = 12
\]
\[
x = 2\sqrt{3}
\]

Now we just need to establish that this value of \( x \) gives us a maximum angle \( a \).

\( a = a(x) \) is a continuously differentiable function on \((0, \infty)\). Note that

\[
\lim_{x \to \infty} a = \lim_{x \to \infty} (\tan^{-1} \frac{6}{x} - \tan^{-1} \frac{2}{x})
\]
\[
= \tan^{-1} 0 - \tan^{-1} 0
\]
\[
= 0
\]
\[
= \frac{\pi}{2} - \frac{\pi}{2}
\]
\[
= \lim_{x \to 0^+} (\tan^{-1} \frac{6}{x} - \tan^{-1} \frac{2}{x})
\]
\[
= \lim_{x \to 0^+} a
\]

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Also, by the monotonicity of arctangent we have that for all $x \in (0, \infty)$, $a(x) > 0$, since

$$\frac{6}{x} > \frac{2}{x} \Rightarrow \tan^{-1} \frac{6}{x} > \tan^{-1} \frac{2}{x}$$

(this is apparent from the physical setup of the problem as well). Thus we have a positive, continuously differentiable function on $(0, \infty)$ that limits to zero on both ends with one critical value. Therefore, the critical value $x = 2\sqrt{3}$ ft, gives us a maximum viewing angle.