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Tangents to Circles

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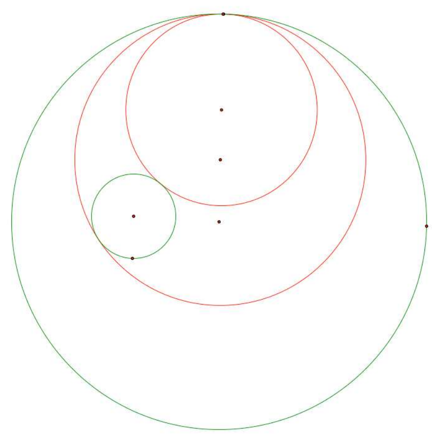
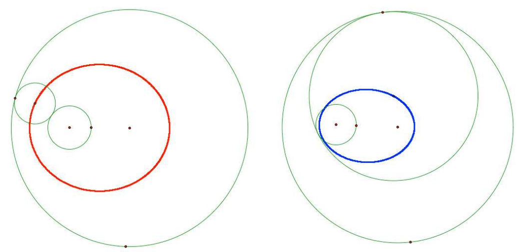
Given two circles, with one completely inside the other, and an arbitrary point on the larger circle, we constructed script tools in GSP to construct a circle tangent to both circles with the arbitrary point being one of the points of tangency. There are two such circles, shown in red, in the following figure.



We could ask what is the locus of the constructed circle centers, as the arbitrary point varies over the large circle. The following figure suggests that in both cases the loci are ellipses. Let's prove this.

Claim: The locus of centers of the constructed circles tangent to two circles with one on the interior of the other and tangent to an arbitrary point on the larger circle, is an ellipse.

Proof. First we consider the case where the constructed circle is externally tangent to smaller circle. Let r_0 be the radius of the larger given circle, r_1 be the radius of the smaller



given circle, r_c be the radius of the constructed circle, and x be the distance from the center of the large circle to the center of the constructed circle. So $r_c + x = r_0$, and thus

$$r_c + r_1 + x = r_c + r_1 + r_0 - r_c = r_1 + r_0$$

which is independent of the arbitrary point and thus for our two given circles is a constant. Thus the sum of the distances from the constructed circle's center to the two given centers is constant and thus the locus of centers is an ellipse.

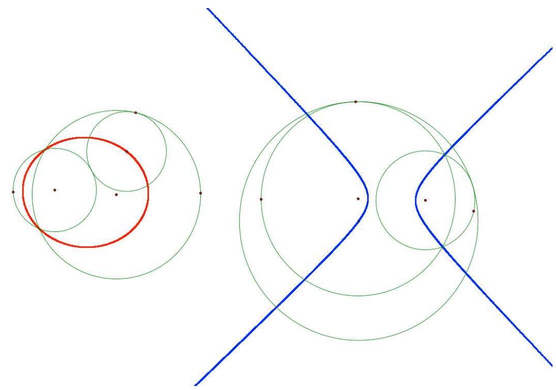
Next we consider the case where the smaller circle is internally tangent to constructed circle. With the same variables as above we again consider the sum of the distances from the constructed circle center to the given circle centers and we have

$$r_c - r_1 + x = r_c - r_1 + r_0 - r_c = r_0 - r_1$$

which is still a constant and thus the locus of centers is an ellipse in this case too. □

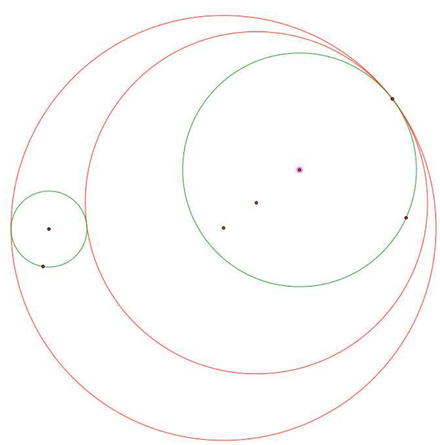
If we allow the two given circles to intersect, then our constructions still work and generate a circle with the required tangencies. The following figure shows the locus of centers for the two constructions.

The first still appears to be an ellipse, and the proof given still holds in this case. For the second construction it appears that the locus of centers is now a hyperbola. We can prove this by re-examing the relationships. Now we have $r_0 + x = r_c$ so that



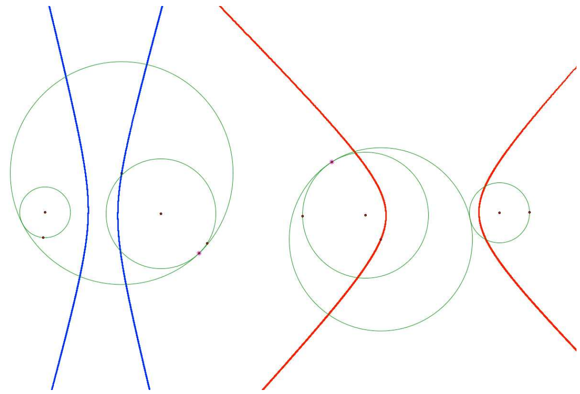
$r_c - r_1 - x = r_c - r_1 + r_0 - r_c = r_0 - r_1$, so the difference of the distances from the constructed circle center to the given circle centers is constant, and the locus is a hyperbola with the two given centers as foci.

We can also construct two different tangent circles when the two given circles are disjoint and both lie exterior to each other. The following graphic shows the two constructed circles in red.



We can trace the locus of the circle centers for both circles and it appears that in each case we have a hyperbola. Let's prove this.

Proof. First let's consider the case where the constructed circle is externally tangent to



the second given circle. Here we have $r_c = x + r_0$, and thus

$$r_c + r_1 - x = r_c + r_1 + r_0 - r_c = r_1 + r_0$$

which is constant. So we do have a hyperbola.

In the second case, where the constructed circle has the second circle internally tangent to it we have

$$r_c - r_1 - x = r_c - r_1 + r_0 - r_c = r_0 - r_1$$

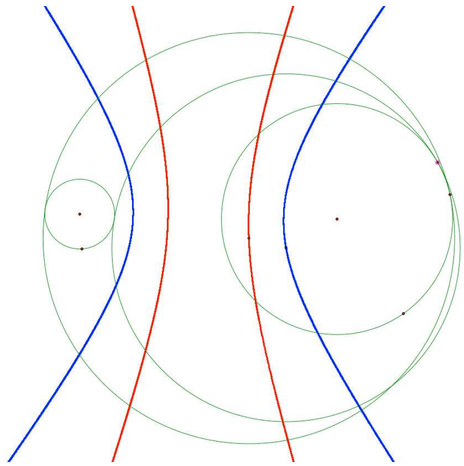
. □

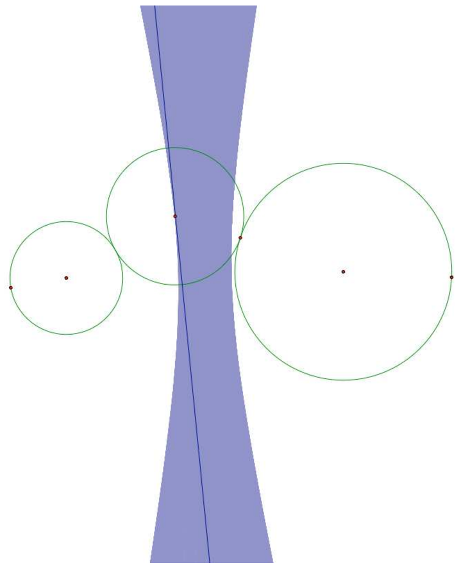
So in both cases the difference of the distances from the constructed center to given centers is constant. Though in the two different cases the constant is different. Thus for the same two given circles the two cases yield two different hyperbola having the same foci.

The next two figures show the traces of the tangent lines to the loci of the centers. These traces exhibit the hyperbolae as an envelope of lines.

By considering some other points to trace we were able to get some interesting curves. The next figure shows the trace of the point that is the midpoint of the point of tangency on the first given circle with the center of the constructed circle.

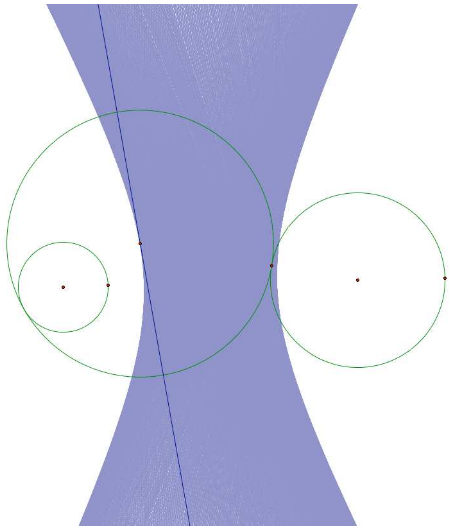
Finally we constructed a script tool to construct two circles tangent to a given circle at an arbitrary point and a line. The construction goes as follows. Let K be the center of the





circle and P be a point on the circle. Let L be a line. Let's suppose that the line through P and K is not perpendicular to L, otherwise there is only one tangent circle (and it is easy to construct). Let M be the line through P, tangent to the circle. By our assumption, M intersects L. Let A be this intersection point. Choose points B and C on L on either side of A. Let D be the point of intersection of the angle bisector of $\angle PAB$ with the line KP. We claim the circle with center at D, passing through P, is one of the required circles. We now prove this claim.

Proof. Let E be the point of intersection of the perpendicular to L through D and L.



Since $\angle PAB$ was bisected, $\angle PAD = \angle DAE$. Also $\angle DPA = \angle DEA$ since both are right angles. Thus ΔPAD is similar to ΔEAD. But since both triangles have the common side AD, they are congruent. Thus the length of PD, is the same as the length of PE. This means that our constructed circle is tangent to L at E. The other circle is gotten similarly by bisecting $\angle PAC$. $\square$

Our final two figures illustrate the proof and show the two constructed circles.

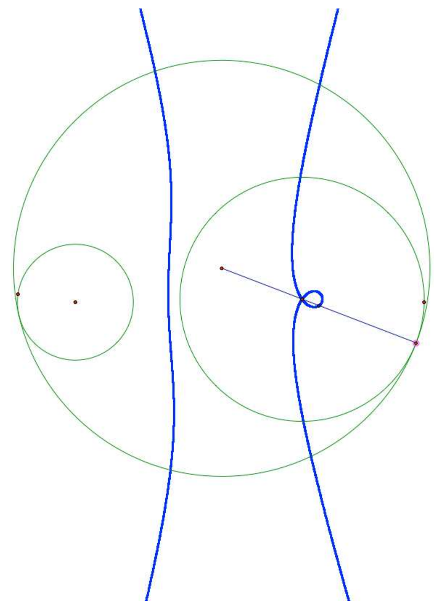


Figure 1: a trace of the midpoint of the given point of tangency with the center of the constructed circle