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Altitudes

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Problem: Let ΔABC be an acute triangle with orthocenter H. Let D, E, and F be the feet of the altitudes through A, B, and C, respectively. Prove that

$$\frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} = 1$$

Solution: Since AD is perpendicular to BC, $\frac{1}{2}(AD)(BC) = Area(\Delta ABC)$ and $\frac{1}{2}(HD)(BC) = Area(\Delta HBC)$. Similarly we have $\frac{1}{2}(BE)(AC) = Area(\Delta ABC)$, $\frac{1}{2}(HE)(AC) = Area(\Delta HAC)$, $\frac{1}{2}(CF)(AB) = Area(\Delta ABC)$, and $\frac{1}{2}(HF)(AB) = Area(\Delta HAB)$.

Thus

$$\begin{aligned}
\frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} &= \frac{2 \cdot Area(\Delta HBC)}{(BC)(AD)} + \frac{2 \cdot Area(\Delta HAC)}{(AC)(BE)} + \frac{2 \cdot Area(\Delta HAB)}{(AB)(CF)} \\
&= \frac{2 \cdot Area(\Delta HBC)}{2 \cdot Area(\Delta ABC)} + \frac{2 \cdot Area(\Delta HAC)}{2 \cdot Area(\Delta ABC)} + \frac{2 \cdot Area(\Delta HAB)}{2 \cdot Area(\Delta ABC)} \\
&= \frac{Area(\Delta HBC) + Area(\Delta HAC) + Area(\Delta HAB)}{Area(\Delta ABC)} \\
&= \frac{Area(\Delta ABC)}{Area(\Delta ABC)} \\
&= 1
\end{aligned}$$

Note that this equation cannot hold if the triangle is obtuse, since (letting $\angle ABC$ be the obtuse angle) $\frac{HE}{BE} > 1$. We tried two other reasonable sums of ratios and neither came out to be 1.