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The Third Pedal Triangle Being Similar to the Original

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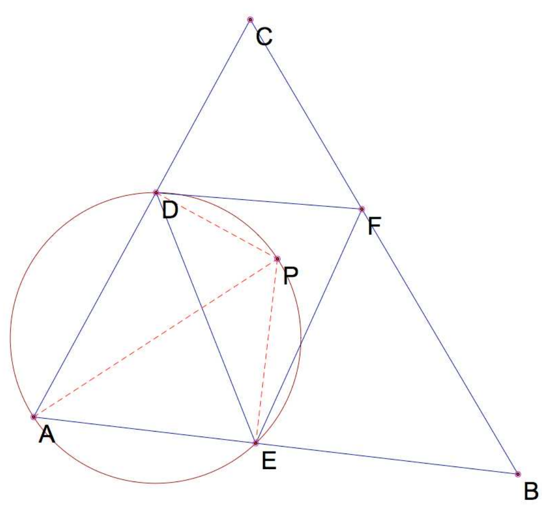
Problem: Given a triangle ΔABC and a pedal point P interior to ΔABC, the third iterated pedal triangle to ΔABC is similar to ΔABC.

Solution:

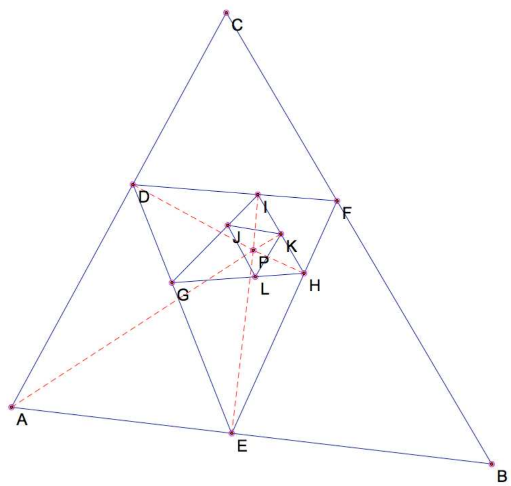
Lemma 1 Let ΔABC be a triangle and P be a point in the interior of ΔABC. Let ΔDEF be the corresponding pedal triangle (see figure). Then $\angle PAB = \angle PDE$.



Proof. Let $\mathcal{C}$ be the circle with $\overline{AP}$ as a diameter. Since both $\angle PDA$ and $\angle PEA$ are right angles and both subtend $\overline{AP}$, points D and E lie on circle $\mathcal{C}$. Since both $\angle PAE$ and $\angle PDE$ are on $\mathcal{C}$ and subtend the segment $\overline{PE}$, $\angle PAB = \angle PDE$.

□

Now we prove the third pedal triangle is similar to the original. Let the original triangle and its three pedal triangles be labeled as in the figure. We are going to apply Lemma 1 repeatedly to show $\angle CAB = \angle JKL$.



Let $\alpha = \angle PAB$. So by Lemma 1 we have

$$\alpha = \angle PAB = \angle PDE = \angle PIG = \angle PKJ$$

Also letting $\beta = \angle PAC$ we have

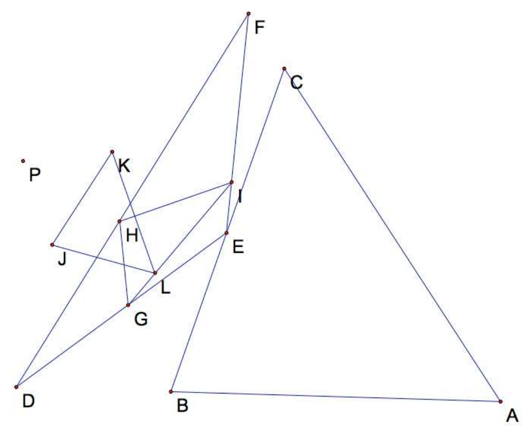
$$\beta = \angle PAC = \angle PED = \angle PHG = \angle PKL$$

So

$$\angle CAB = \angle PAB + \angle PAC = \alpha + \beta = \angle PKJ + \angle PKL = \angle JKL$$

Problem: Given a triangle ΔABC and a pedal point P exterior to ΔABC (but not on the circumcircle to ΔABC), the third iterated pedal triangle to ΔABC is similar to ΔABC.

Solution: We insist that P is not on the circumcircle to ΔABC so that the pedal triangle does not degenerate. We will also assume that none of the further pedal triangles degenerate and argue based on our figure. We will relate the angles in each triangle to the angles in the respective pedal triangle.



We have a circle through $PEBD$, denoted by $O(PEBD)$, since $\angle PEB$ and $\angle PDB$ are right angles, thus $\angle CBP = \angle EDP$. Also $\angle DEP = \angle DBP$ so that $\pi = \angle ABP + \angle DBP = \angle ABP + \angle DEP$. By noting that we have a circle $O(PFCE)$, we have $\pi = \angle ACP + \angle FCP = \angle ACP + \angle FEP$. Also $\angle BCP = \angle EFP$. By looking at $O(PFAD)$, we have $\angle CAP = \angle FDP$ and $\angle BAP = \angle DFP$.

Now we move in one triangle. Since $\angle EDP$ and $\angle GHP$ subtend supplementary arcs of $O(PHGD)$, $\angle EDP + \angle GHP = \pi$. From the same circle we have $\angle FDP = \angle HGP$. Due to $O(PGIE)$, we have $\angle DEP = \angle GIP$ and $\angle FEP = \angle IGP$. From $O(PHIF)$, $\angle EFP + \angle IHP = \pi$ and $\angle DFP = \angle HIP$.

From $O(PJHK)$, we have $\pi = \angle GHP + \angle KHP = \angle GHP + \angle KJP$, and $\pi = \angle IHP + \angle JHP = \angle IHP + \angle JKP$. From $O(PJLI)$, we get $\pi = \angle GIP + \angle LJP$ and $\angle HIP = \angle JLP$. Finally from $O(PKLG)$, we get $\pi = \angle IGP + \angle LKP$, and $\angle HGP = \angle KLP$.

Putting everything together we have

$$
\begin{aligned}
\angle CAB &= \angle CAP + \angle BAP \\
&= \angle FDP + \angle DFP \\
&= \angle HGP + \angle HIP \\
&= \angle KLP + \angle JLP \\
&= \angle KLJ
\end{aligned}
$$

Also

$$
\begin{aligned}
\angle ABC &= \angle ABP - \angle CBP \\
&= \pi - \angle DEP - \angle EDP \\
&= \angle GHP - \angle GIP \\
&= \pi - \angle KJP - (\pi - \angle LJP) \\
&= \angle LJP - \angle KJP \\
&= \angle LJK
\end{aligned}
$$

And finally

$$
\begin{aligned}
\angle ACB &= \angle ACP - \angle BCP \\
&= \pi - \angle FEP - \angle EFP \\
&= \angle IHP - \angle IGP \\
&= \pi - \angle JKP - (\pi - \angle LKP) \\
&= \angle LKP - \angle JKP \\
&= \angle LKJ
\end{aligned}
$$

Thus ΔABC is similar to ΔJKL.