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A Rope, a Goat, a Shed, and a Silo

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Problem: Grazing area.

Farmer Jones had a goat on a tether. He tied the end of the tether not attached to the goat to a stake in a field. Over what area could the goat graze? Of course you need to know something about the length of the tether and about the field.

There are two structures in the field:

– a shed that is 20 ft long and 20 feet wide (square)

– a silo that is 20 ft in diameter

The center of the shed and the center of the silo are on a line and the distance apart is 92 feet. The distance from center to center, if you wanted to use this data, is 112 feet.

The tether for the goat is 76.7 feet long. The stake to which the tether is tied is somewhere along the line of centers between the shed and the silo.

Explore the area over which the goat can graze as the stake is moved along this line segment from the midpoint of the side of the shed to the edge of the silo.

Solution:

We will explore the area the goat can graze over in several cases. We have included most of the graphics, at least one for each case, at the end of the document. Also note that we only reference the labels in the graphics for case 1 and case 2a. In the other graphics the labels are not consistent and we never reference them in this document. Also note that the figures without numbers are not to scale, whereas the figures with numbers are to scale, and in fact our calculations are based on those numbers. A final caveat is that all angles referenced in the body of the document are in radians (and thus unitless). The angles obtained from GSP are shown in degrees in the figures, but we converted them to radians in the document.

Case 1:

First we'll consider the case when the stake is placed very near to the shed, so that the goat cannot reach the silo. Let z be the distance the stake is placed from the wall of the shed. Since we don't want the goat to be able to reach the silo, this case considers $z \in [0, 15.3]$. Let $\theta = \angle BAC = \tan^{-1} 10/z$. So $AB = \sqrt{z^2 + 10^2}$. Note that for $z = 15.3$, $AB = 18.3$, and the goat still has 58.4 feet of tether at B, which exceeds the 30 feet minimum he would need to reach point E.

So the goat can range over a sector of a circle with maximal radius (equal to the length of the tether) and angle $2\pi - 2\theta$. This contributes an area of

$$1/2 \cdot (2\pi - 2\theta)(76.7)^2 = (\pi - \tan^{-1} 10/z)(76.7)^2$$

There is also a triangle on the near side of the shed of area

$$10z$$

The sector of the circle at B of grazing field has area $1/2 \cdot \theta(76.7 - \sqrt{z^2 + 100})^2 = 1/2 \cdot (\tan^{-1} 10/z)(76.7 - \sqrt{z^2 + 100})^2$, but there are two of these contributing an area of

$$\theta(76.7 - \sqrt{z^2 + 100})^2 = (\tan^{-1} 10/z)(76.7 - \sqrt{z^2 + 100})^2$$

When the goat is at point D he has $76.7 - \sqrt{z^2 + 100} - 20 = 56.7 - \sqrt{z^2 + 100} = DE$ feet of tether left. If we let the angle of the circular sector at D be γ, then the area due two the two circular sectors at the back corners of the shed is

$$\gamma(56.7 - \sqrt{z^2 + 100})^2 = \sin^{-1} \frac{10}{56.7 - \sqrt{z^2 + 100}} \cdot (56.7 - \sqrt{z^2 + 100})^2$$

The last bit of grazing area comes from ΔDEF and its reflection. This area is

$$2\Delta DEF = 10 \cdot \sqrt{(56.7 - \sqrt{z^2 + 10^2})^2 - 10^2}$$

Summing these gives the grazing area as a function of z

$$Area = (\pi - \tan^{-1} 10/z)(76.7)^2 + 10z + (\tan^{-1} 10/z)(76.7 - \sqrt{z^2 + 100})^2$$

$$+ \sin^{-1} \frac{10}{56.7 - \sqrt{z^2 + 100}} \cdot (56.7 - \sqrt{z^2 + 100})^2 + 10 \cdot \sqrt{(56.7 - \sqrt{z^2 + 10^2})^2 - 10^2}$$

We used Matlab to plot this function. As we see it increases in z, thus the maximal area for this case occurs with $z = 15.3$ and we have an area of 17965 square feet.

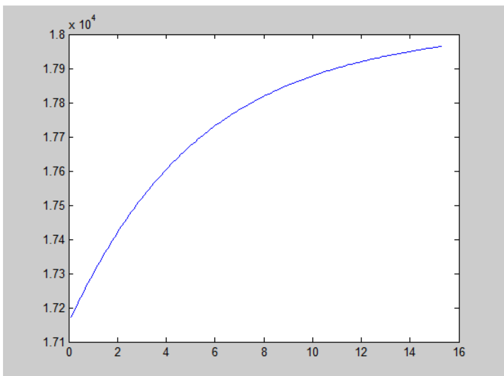

Figure 1: graph of area vs. z for case 1

As the stake is placed closer to the silo, the grazing area initially must decrease. This is because we are losing area on the back side of the shed as well as the silo starts getting in the way.

Case 2:

Let's examine the case when the goat can just reach the side of the silo at point G. We used a GSP construction to figure out where the stake should go so that the goat can just reach point G. The picture is roughly shown below. (We say roughly because in the figure we ignored the fact that there is a small piece of the tether that coincides with the silo.) The stake should be placed about 66.03 feet from the wall of the silo.

We will use the same labels as in case 1. Also let the angle formed by G, A, and the center of the silo be φ. So $\varphi = \tan^{-1} 10/76.03 = 0.1308$. Also $z = 92 - 66.03 = 25.97$ feet. So $\theta = \tan^{-1} 10/25.97 = 0.3676$. Thus the area over which the goat has maximal tether is

$$1/2 \cdot (2\pi - 2\theta - 2\varphi)(76.7)^2 = 15550$$

Let's quickly deal with the areas around the shed. Since we did them in general in case 1, we can do them with numbers for case 2. The triangle on the front side of the shed has area

$$10(25.97) = 259.7$$

The goat has 48.87 feet of tether at point B so that twice the area of the sector at B is

$$\theta r^2 = 0.3676(48.87)^2 = 878$$

At point D there is 28.87 feet of tether. So angle $\gamma = \sin^{-1} 10/28.87 = 0.3537$, so twice the area of the sector at D is

$$\gamma r^2 = 0.3537(28.87)^2 = 295$$

$EF = \sqrt{28.87^2 - 100} = 27.08$ so that

$$2 \cdot \Delta DEF = 10(27.08) = 271$$

Now to deal with the portion of the grazing area near the silo. We can consider the triangle formed by A, G, and the antipodal point to G. Half of the area of the disk of the silo eats into this triangle. So the goat has about another

$$10(76.03) - \frac{1}{2}\pi(10)^2 = 603$$

square feet of grazing area.

So all in all this configuration gives the goat about

$$17860 \text{ sq ft}$$

of grazing area.

Case 3:

Let's consider the case where the goat can just reach the far side of the silo. We used GSP to see where the stake should be placed and found that it should be 50.2 feet from the wall of the silo, so that $z = 41.8$. We obtained the angle φ from GSP as well and found it to be 0.167. Also we compute $\theta = \tan^{-1} 10/41.8 = 0.2348$. Thus we have a maximally tethered grazing area of

$$(\pi - \theta - \varphi)(76.7)^2 = 16120$$

Let's next deal with the area near the silo (meaning the region over which the goat can graze where rays from the silo to the points intersect the silo). We will approximate this as a sector of a circle minus the disk of the silo. The angle of the sector will be 2φ. The radius we will use is going to be the average of the shortest radius and the longest radius. We will use as the shortest radius the distance between the stake and the furthest point the goat can reach on the centerline behind the silo. The longest radius will always be the full tether length, 76.7. So in this case the shortest radius is 70.2 feet, and the average radius is

73.5 feet. So the area associated to the silo is

$$\varphi(73.5)^2 - \pi(10)^2 = (0.167)(73.5)^2 - \pi(10)^2 = 588$$

The other computations involving the shed are as in the previous two cases and we will omit the details and just list the values associated with each region. It should be noted that also in this case the goat can still reach the centerline behind the shed, with 3.7 feet of tether to spare. The triangle on the front side of the shed has area 418, the two sectors at B have area 267, the two sectors at D have area 153.6, and twice the area of ΔDEF is 93.6. Yielding a total grazing area of about 17640 sq feet.

Case 4:

The next case we will examine is when the goat can just reach the centerline behind the shed. So 30 feet of tether is used up from that point to B, meaning that $AB = 46.7$. Thus Pythagoras yields $z = 45.6$, and the stake is 46.4 feet from the wall of the silo. We then set up a GSP construction to see how far the goat can reach along the center line behind the silo and the angle φ. We found that the goat's farthest distance along the centerline behind the silo from the stake is 72.7 feet. We use the same method of approximating the area associated to the silo as in the previous case. Thus we compute an area of

$$\varphi\left(\frac{1}{2}(72.7 + 76.7)\right)^2 - \pi(10)^2 = 681$$

The other computations are similar to the ones before. The maximal tethered region has area 16160, the front triangle of the shed is 456, the two sectors at B are 194, and the two sectors at D are 157, yielding a total area of 17650.

Case 5:

This case will be where the goat can just reach point D. The methods in this case are identical to the previous including the treatment of the area associated with the silo. GSP gave us $\varphi = 0.218$, and a shortest radius of 73.8 feet on centerline behind the silo, yielding an average radius behind the silo of 75.3 feet. Note that the variation in the radius is not very large thus we feel this is a very good approximation to the actual area obtained. The numbers came out as follows. The silo area was 922; the maximal tethered was 16160; the front shed triangle was 558, and two sectors at B were 70.9. The total area is thus 17710 sq feet.

Case 6:

This case will be where the goat can just reach point B. GSP gave us a shortest radius of 73.7 feet on centerline behind the silo, yielding an average radius behind the silo of 75.2 feet. Also $\varphi = 0.394$ (from GSP). The numbers worked out as follows. The silo area was 1914; the maximal tethered was 15394; and the front shed triangle was 760. The total area is thus 18070 sq feet.

Case 7:

This case will be where the goat can just reach the shed at one point, point C. GSP gave us a shortest radius of 73.7 feet on centerline behind the silo, yielding an average radius behind the silo of 75.2 feet. Also $\varphi = 0.407$ (from GSP). The numbers worked out as follows. The silo area was 1987; and the maximal tethered was 16090. The total area is thus 18080 sq feet.

Case 8:

Our last case will explore when the stake is at the front wall of the silo. From GSP we obtained a distance from the stake on the far side of the silo of 70.2 feet, giving an average radius of 73.5 ft. Of course, in this case $\varphi = \pi/2$. So the area associated to the silo is $\frac{1}{2}\pi(73.5)^2 - \pi(100) = 8170$, and the maximally tethered semicircle has area $\frac{\pi}{2}(76.7)^2 = 9241$, yielding a total area of 17410 sq ft.

So in conclusion, by our estimates if the farmer wants to have his goat have the maximal amount of area for grazing, he should stake the goat so that the goat can just touch the shed, i.e. he should go with case 7.

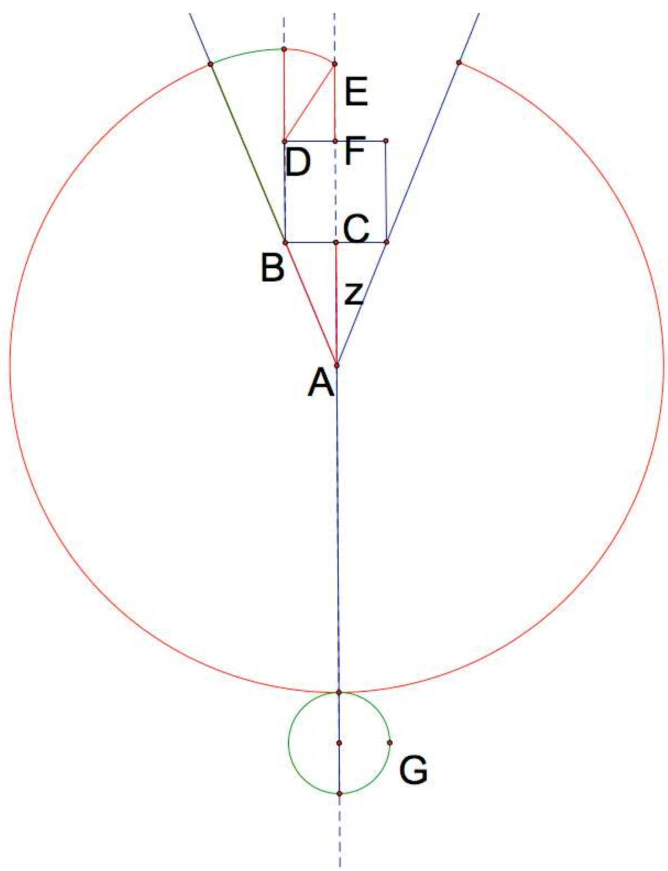

Figure 2: Case1

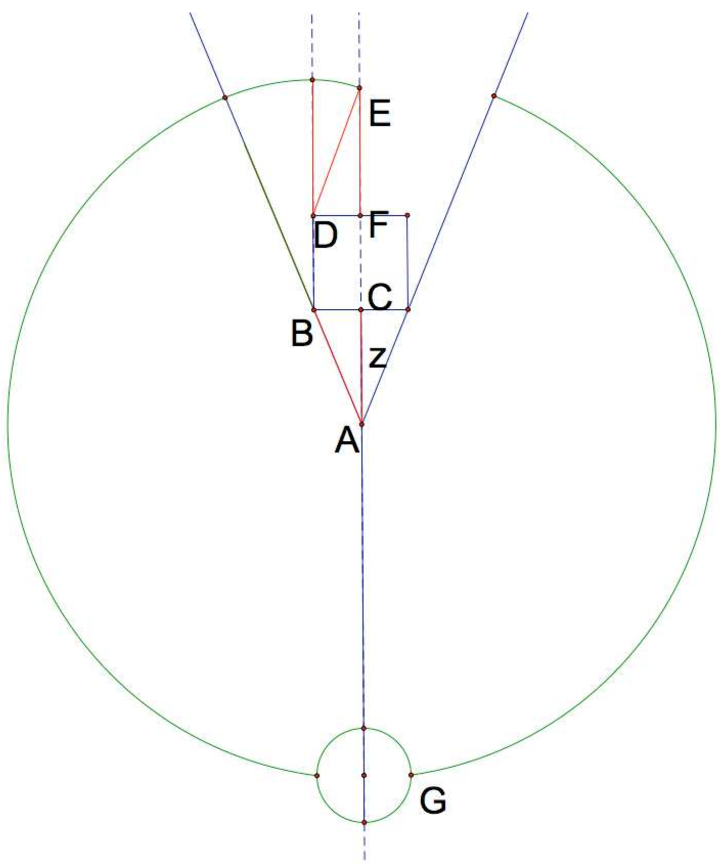

Figure 3: Case2a

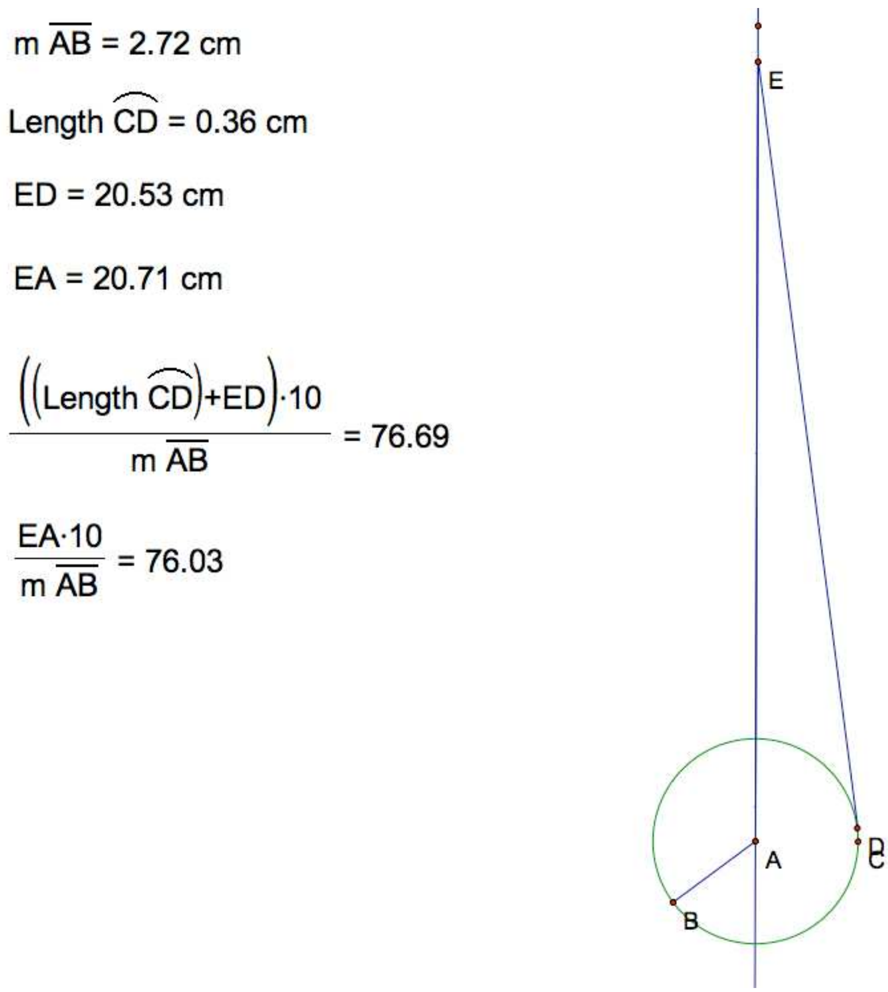

Figure 4: Case2b

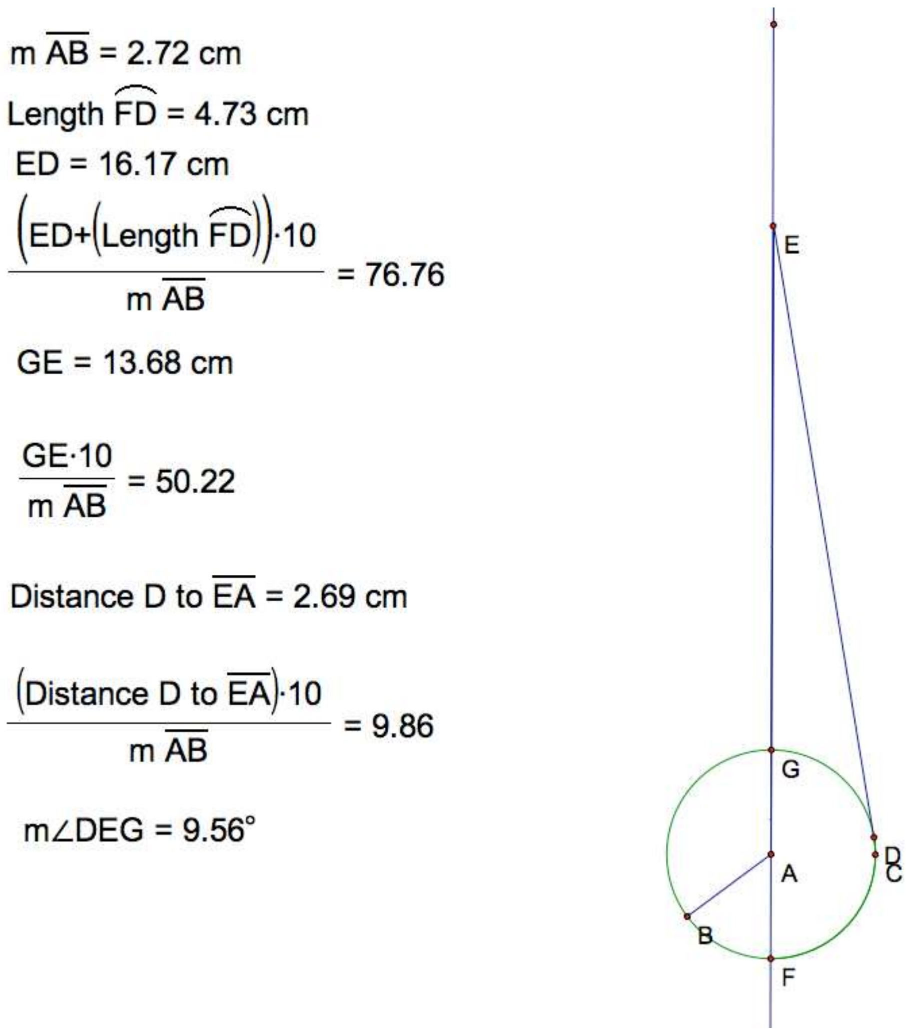

Figure 5: Case3

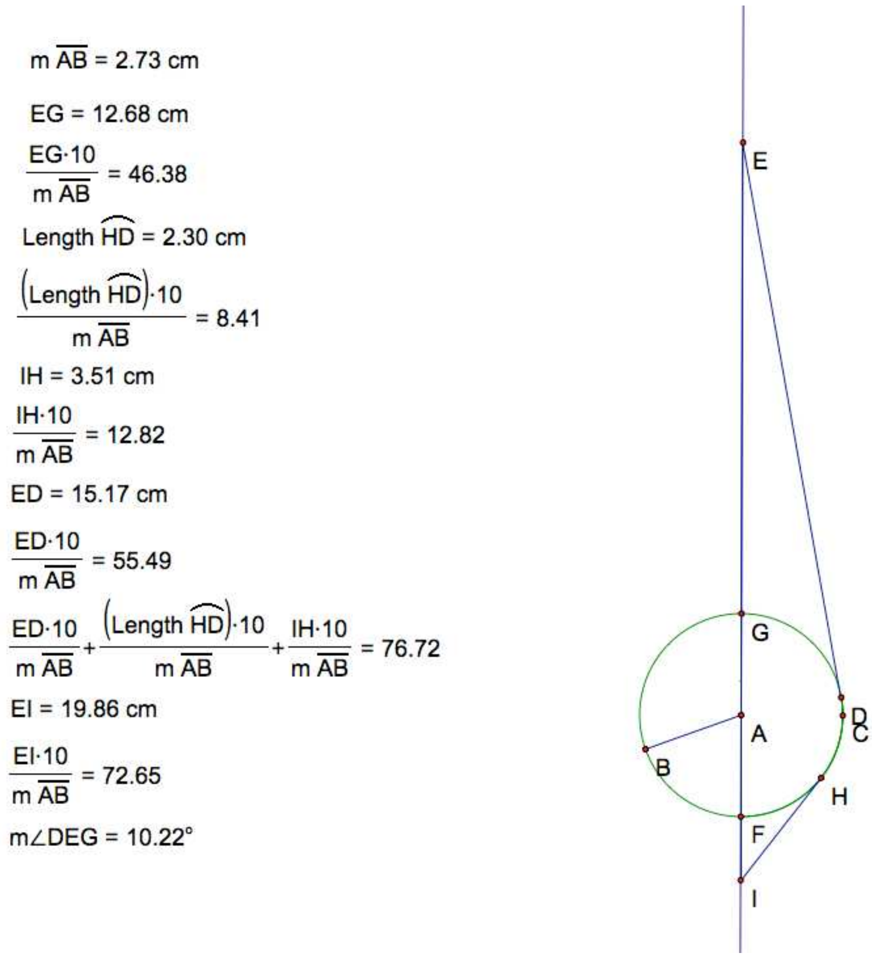

Figure 6: Case4

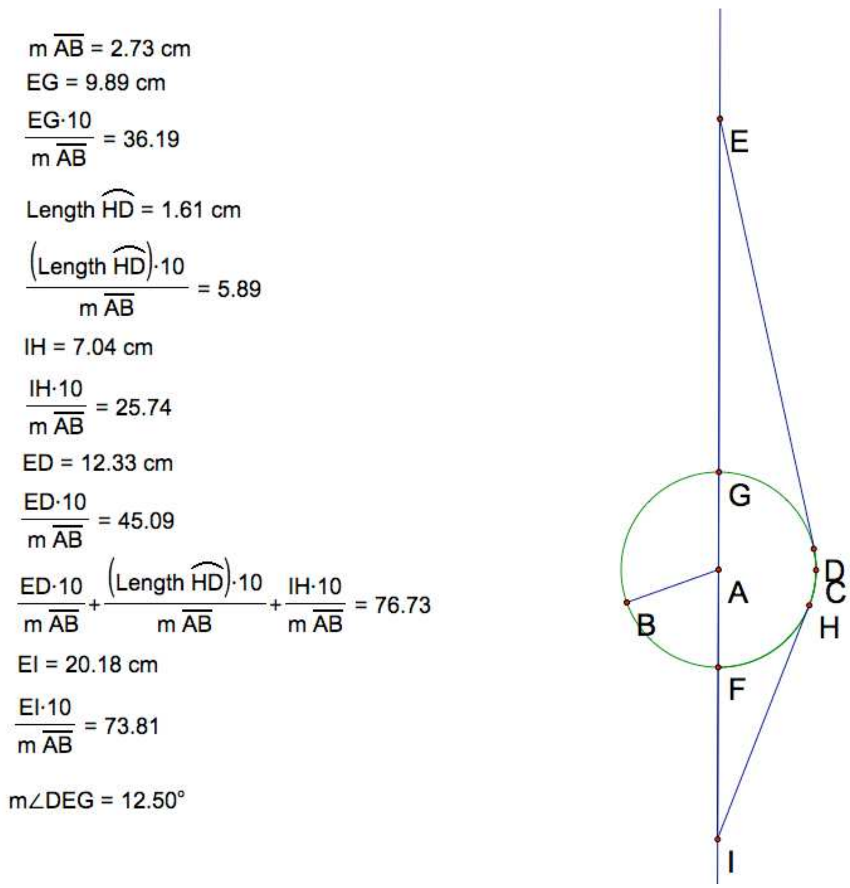

Figure 7: Case5

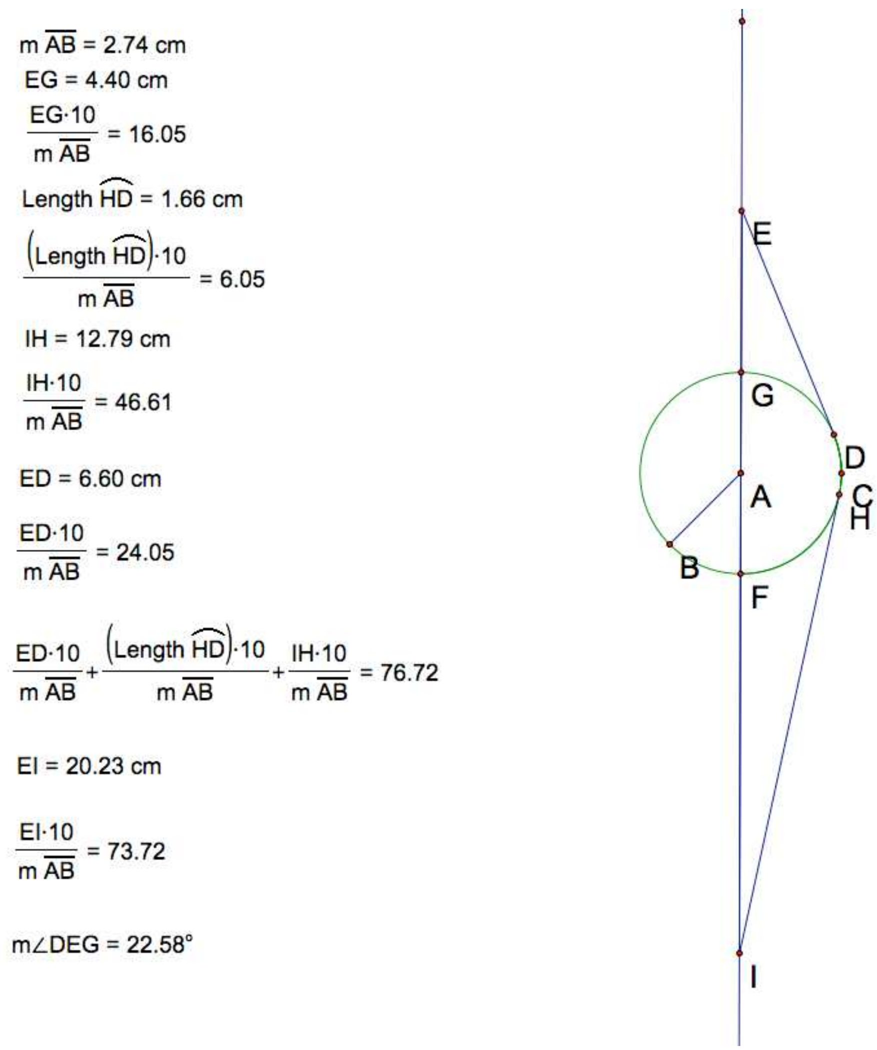

Figure 8: Case6

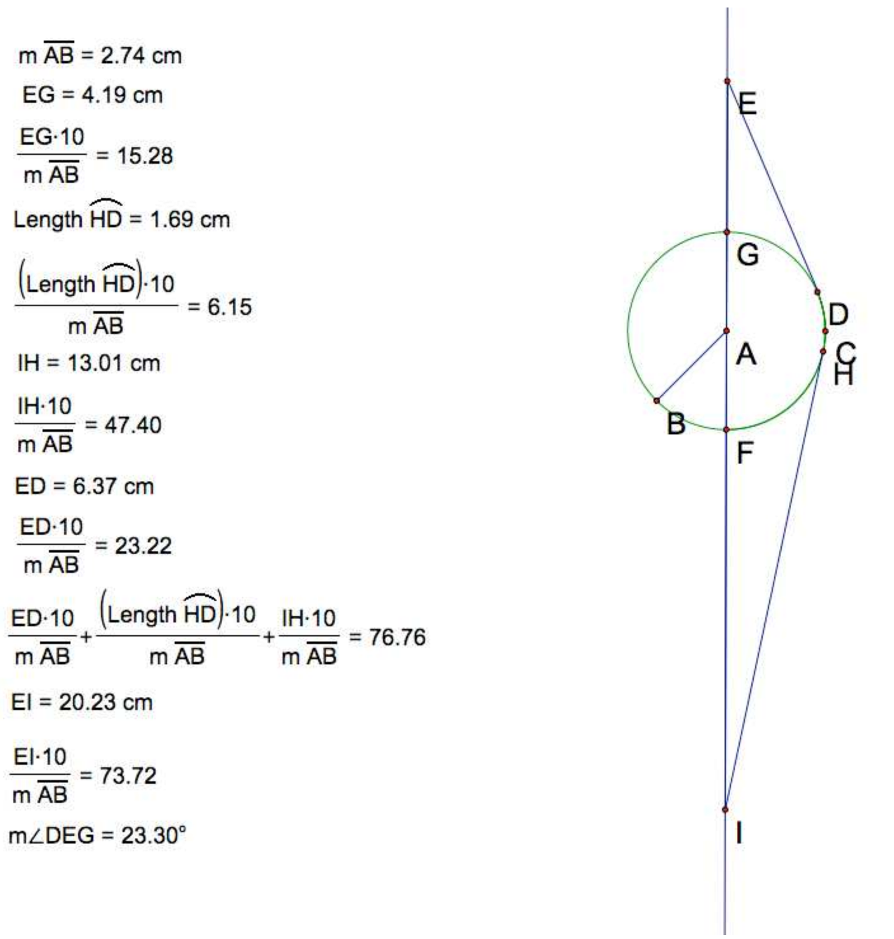

Figure 9: Case7

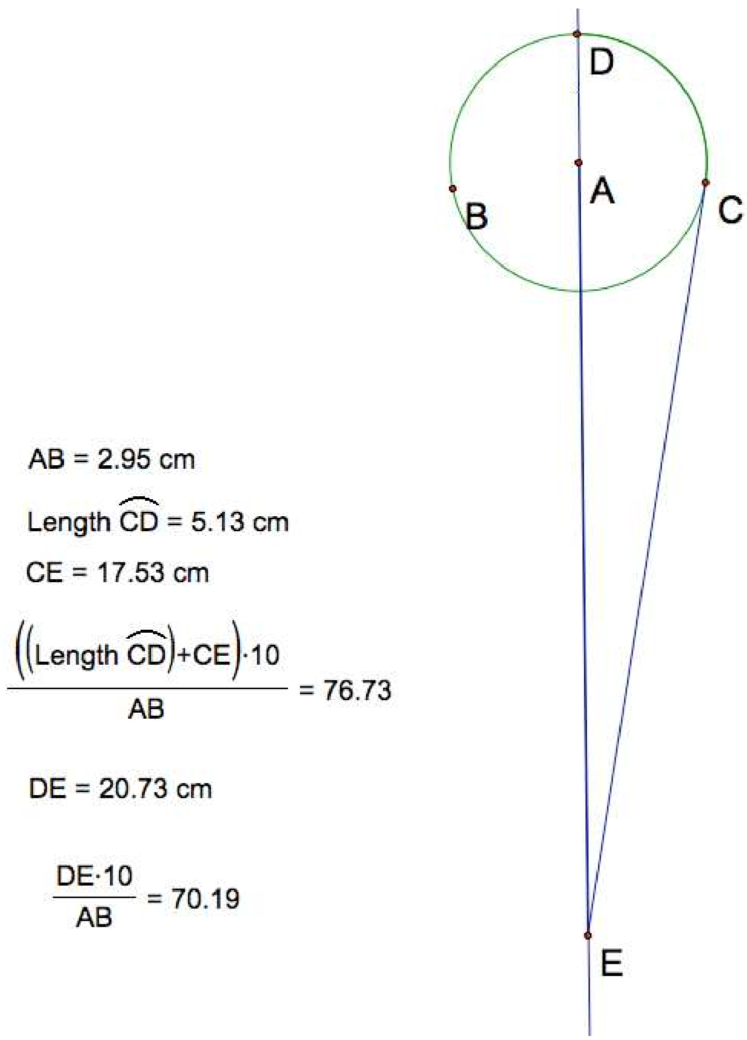

Figure 10: Case8