In this assignment, we look at the sum of the ratios of side lengths formed by connecting vertices to the orthocenter.

I. Given triangle ABC, construct the orthocenter H. Let points D, E and F be the feet of the perpendiculars from A, B and C, respectively. Prove:

\[
\frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} = 1 \quad \text{and} \quad \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2
\]

Given any acute triangle ABC, construct the altitudes and take the feet of the altitudes to be D, E and F. Let the intersection of the altitudes (i.e. the orthocenter), be called H. While it is not relevant to the proof given, it’s interesting to observe the similarity of triangles:

- \( AFH \sim CDH \) (angle AHF = angle CHD because they are vertical angles, angle HFA = angle HDC = right angles, and so we get equality of the third angles)
- \( AEH \sim BDH \) (angle AHE = angle BHD because they are vertical angles, angle HEA = angle HDB = right angles, and so similarity follows)
- \( EHC \sim FHB \) (angle EHC = angle FHB because they are vertical angles, angle HEC = angle HFB = right angles, so third angles are equal)

Now by similar triangles, we have proportionality of sides:

<table>
<thead>
<tr>
<th></th>
<th>( \triangle AFH \sim \triangle CDH )</th>
<th>( \triangle AEH \sim \triangle BDH )</th>
<th>( \triangle EHC \sim \triangle FHB )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{AH}{CH} )</td>
<td>( \frac{HF}{HD} ) = ( \frac{AF}{CD} )</td>
<td>( \frac{EH}{BH} ) = ( \frac{HE}{HD} = \frac{AE}{BD} )</td>
<td>( \frac{CH}{BH} ) = ( \frac{EH}{FH} = \frac{EC}{FB} )</td>
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Now it is clear that \( \frac{AH+HD}{AD} = 1 \) and \( \frac{EH+HB}{EB} = 1 \) and \( \frac{CH+HF}{CF} = 1 \).

So adding these three equations, splitting the fractions and rearranging some of the terms, we see

\[
\left( \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} \right) + \left( \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} \right) = 3.
\]

Now to see \( \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} = 1 \), we compare areas of sub-triangles to the big triangle:

\[
\frac{A_{BHC}}{A_{BAC}} = \frac{1}{2} \frac{BC \cdot HD}{BC \cdot AD} = \frac{HD}{AD}
\]

\[
\frac{A_{CHA}}{A_{CBA}} = \frac{1}{2} \frac{AC \cdot HE}{AC \cdot BE} = \frac{HE}{BE}
\]

\[
\frac{A_{BHA}}{A_{BCA}} = \frac{1}{2} \frac{AB \cdot HF}{AB \cdot CF} = \frac{HF}{CF}
\]
So summing the ratios \[ \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} = \frac{A_{BHC}}{A_{BAC}} + \frac{A_{CHA}}{A_{CBA}} + \frac{A_{BHA}}{A_{BCA}} = \frac{(A_{BHC} + A_{CHA} + A_{BHA})}{A_{ABC}} = \frac{A_{ABC}}{A_{ABC}} = 1, \] as desired. From this and the previously established formula, we get immediately that \[ \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2. \]

**Exploration**

Does this still hold if triangle ABC is obtuse?

No, it does not, as the following example demonstrates.

\[
\begin{align*}
\text{m } \overline{HD} &= 3.49 \text{ cm} \\
\text{m } \overline{AD} &= 2.51 \text{ cm} \\
\text{m } \overline{HE} &= 5.34 \text{ cm} \\
\text{m } \overline{BE} &= 1.42 \text{ cm} \\
\text{m } \overline{HF} &= 3.48 \text{ cm} \\
\text{m } \overline{CF} &= 2.53 \text{ cm} \\
\text{m } \overline{AH} &= 6.00 \text{ cm} \\
\text{m } \overline{BH} &= 3.92 \text{ cm} \\
\text{m } \overline{CH} &= 6.01 \text{ cm}
\end{align*}
\]

\[
\begin{align*}
\frac{\text{m } \overline{HD}}{\text{m } \overline{AD}} + \frac{\text{m } \overline{HE}}{\text{m } \overline{BE}} + \frac{\text{m } \overline{HF}}{\text{m } \overline{CF}} &= 6.54 \\
\frac{\text{m } \overline{AH}}{\text{m } \overline{AD}} + \frac{\text{m } \overline{BH}}{\text{m } \overline{BE}} + \frac{\text{m } \overline{CH}}{\text{m } \overline{CF}} &= 7.54
\end{align*}
\]