The University of Georgia

Mathematics Education Program

J. Wilson, EMAT 6600

Bouncing Barney.

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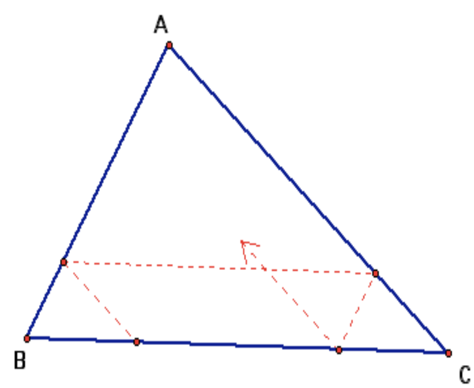
Goal : To investigate the distance Barney travels as he bounces inside and outside of a triangular room.

Problem

Barney is in the triangular room shown here. He walks from a point on BC parallel to AC. When he reaches AB, he turns and walks parallel to BC. When he reaches AC, he turns and walks parallel to AB.



Prove that Barney will eventually return to his starting point.

How many times will Barney reach a wall before returning to his starting point?

Explore and discuss for various starting points on line BC, including points **exterior** to segment BC.

Discuss and prove any mathematical conjectures you find in the situation.

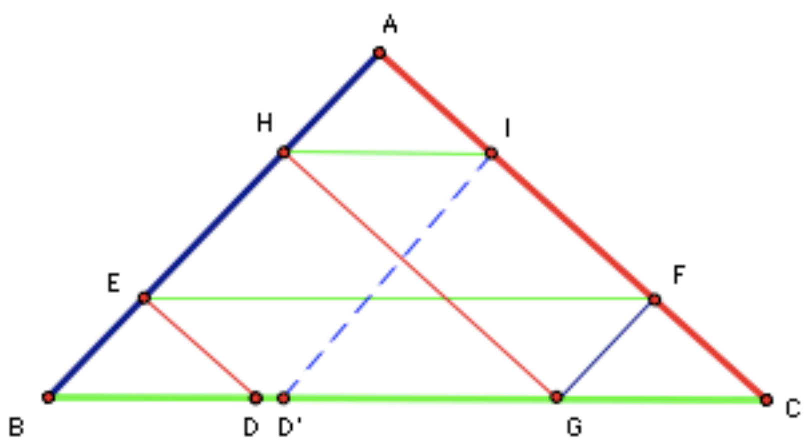
Solution

Conjectures.

1. Start Point is the same as End Point

Proof

Barney starts at point D and bounces parallel to side AC to point E. then bounces parallel to side BC to point F, then parallel to AB to point G, parallel to side AC to point F



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parallel to side BC to point I and finally parallel to side AB to point D'.

Our aim here is to prove that the point D and point D' are the same.

In the diagram above, the parallel lines (segments) are of the same color.

We have the following relationships:

Δ BED ~ Δ ABC, Δ BHG ~ Δ ABC, Δ AHI ~ Δ ABC, Δ AEF ~ Δ ABC, Δ CFG ~ Δ ABC, Δ CID' ~ Δ ABC

use the following theorem and identify proportional segments as Barney bounces along.

Theorem.

If a line is drawn parallel to one side of a triangle intersecting the other two sides, then the other two sides are divided in the same ratio.

Assume point D $\neq$ D'

In step 1: Along path DE

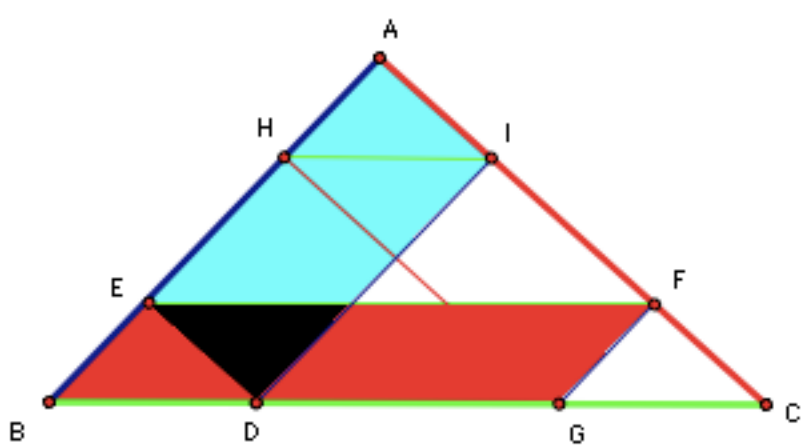
We have: $\frac{BD}{BC} = \frac{BE}{BA}$ In step 2 along path EF we have: $\frac{BE}{BA} = \frac{CF}{CA}$. In step 3 along path FG We have: $\frac{CF}{CA} = \frac{CG}{GB}$ In step 4 along path GH We have: $\frac{CG}{GB} = \frac{AH}{HB}$. In step 5 along path GH We have: $\frac{AH}{HB} = \frac{AI}{IC}$. In step 6 along path HD' , we have: $\frac{AI}{IC} = \frac{BD'}{BC}$.

So looking at the proportionalities so far then we cab see that $\frac{BD}{BC} = \frac{BD'}{BC}$. Hence BD = BD' and thus D = D', a contradiction to the assumption that D is not equal to D'.

So the Start point and End point of Barney's path are the same.

2. The length of Barney's path equals the perimeter of ΔABC, when he starts anywhere other than the midpoint or one of the vertices of side BC of ΔABC.

Proof:

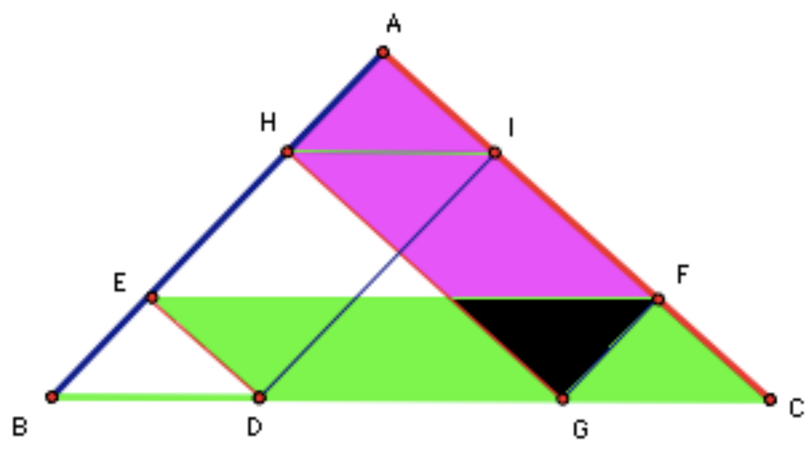


Consider BEFG and AEDI are parallelograms

EA ≅ ID by opposite sides of parallelogram are congruent

EB ≅ FG by opposite sides of parallelogram are congruent

AB = AE + EB = DI + FG

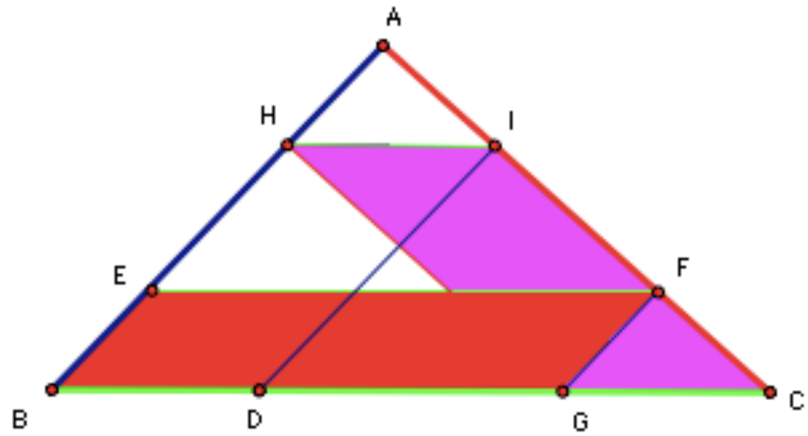


Consider AHFG and EDFC are parallelograms

FA ≅ GH by opposite sides of parallelogram are congruent

CF ≅ DE by opposite sides of parallelogram are congruent

AC = FA + CF = GH + DE



Consider BEFG and IHGC are parallelograms

BG ≅ EF by opposite sides of parallelogram are congruent

GC ≅ HI by opposite sides of parallelogram are congruent

BC = BG + GC = EF + HI

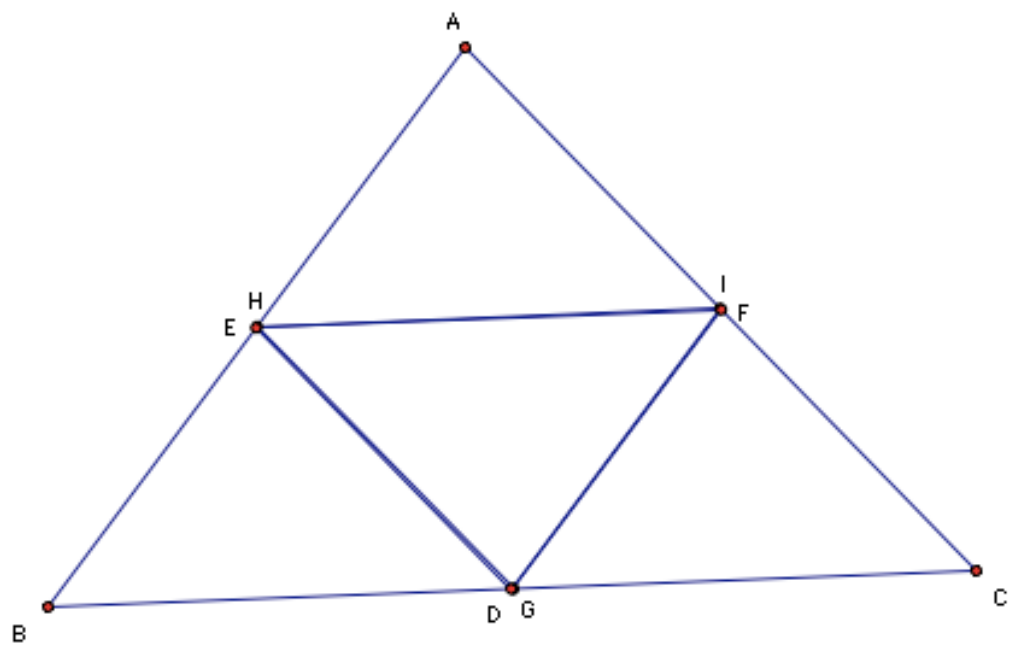
Now the perimeter of the triangle ABC = AB + BC + AC = DI + FG + GH + DE + EF + HI

And the path travelled by Barney = DE + EF + FG + GH + HI + DI = DI + FG + GH + DE + EF + HI = AB + BC + AC, the perimeter of the triangle ABC.

Hence the distance travelled by Barney is equal to the perimeter of the triangle

3. If the Start point is the midpoint of one side of the triangle then the distance that travelled to the End point is one half of the perimeter of ΔABC (or Barney has to travel around twice for the distance travelled to be equal to the perimeter of the to ΔABC).

Proof:

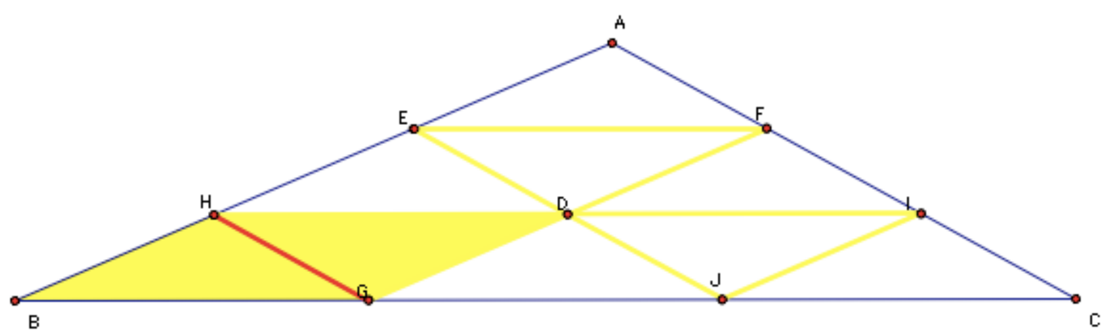


Starting at point D and travelling parallel to AC to point E, then parallel to side AC to point F, forms parallelogram DEIC. Now CD = EF , so BC = 2EF. FC = DE so AC = 2FC. Then travelling parallel to side AB to point G (same as original point D), results in parallelogram EAFG. Now AE = FG, so AB = 2FG.

Thus the perimeter of ΔABC = AB + AC + BC = 2FG + 2FC + 2EF = 2(FG + FC + EF) = twice distance travelled by barney from stat point to end point. In other words Barney has to travel around twice to travel distance equal to the perimeter of ΔABC.

4. If Barney starts at the Centroid (labeled D in the diagram below):

a) Nine congruent triangles are formed. Consider BHDG is a parallelogram. The path GH is a diagonal of the parallelogram and divides the parallelogram into two congruent triangles. Hence $\Delta BHG \approx \Delta DGH$



Using the same reasoning an the following parallelograms:

ii) GHDJ $\Rightarrow \Delta DGH \approx \Delta GJD$

iii) GDIJ $\Rightarrow \Delta GJD \approx \Delta IDJ$

iv) JDIC $\Rightarrow \Delta IDJ \approx \Delta JIC$

v) JDFI $\Rightarrow \Delta IDJ \approx \Delta DIF$

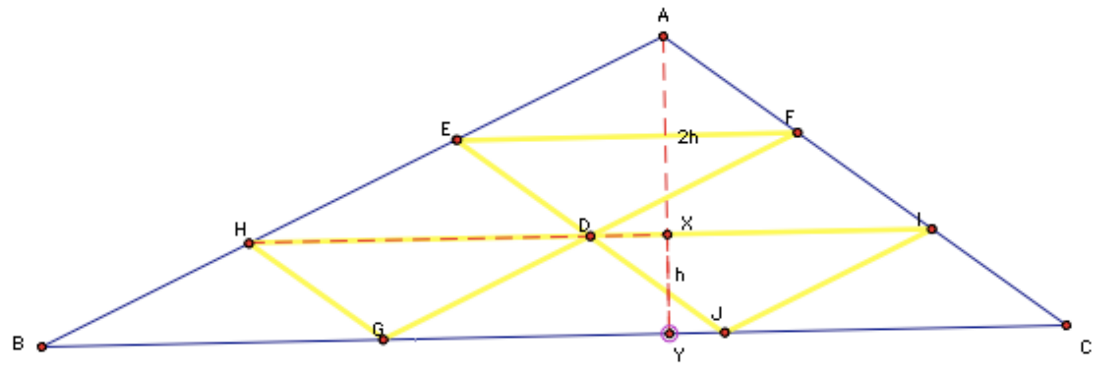
vi) EDIF $\Rightarrow \Delta DIF \approx \Delta FED$

vii) EDFA $\Rightarrow \Delta FED \approx \Delta EFA$

viii) EHDF $\Rightarrow \Delta FED \approx \Delta HDE$

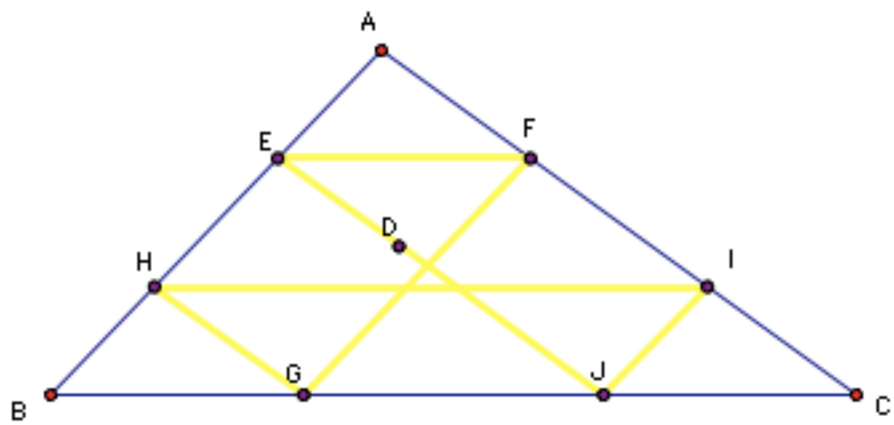
Thus there are 9 congruent triangles.

b) The parallel lines trisect the sides of the ΔABC



Drop a perpendicular from the vertex A to side BC and label the foot of the altitude Y. Let x be the point where the perpendicular cuts the segment HI the first parallel segment from base BC. Let h be the distance between the BC and HI. Now h is the height of ΔDGJ, one of the 9 congruent triangles. Observe that the triangle FED is has its height between the parallels HI and FE. Finally, observe that the perpendicular distance from point A to the parallel EF is the height of triangle EFA. thus the distance from point A to the parallel HI is the vertical height of two of the congruent triangles and thus is twice the distance from the segment HI the first parallel segment to base BC Thus the parallel lines trisect the sides of ΔABC.

5. If Barney starts at the Incenter (labeled as D in the diagram below.

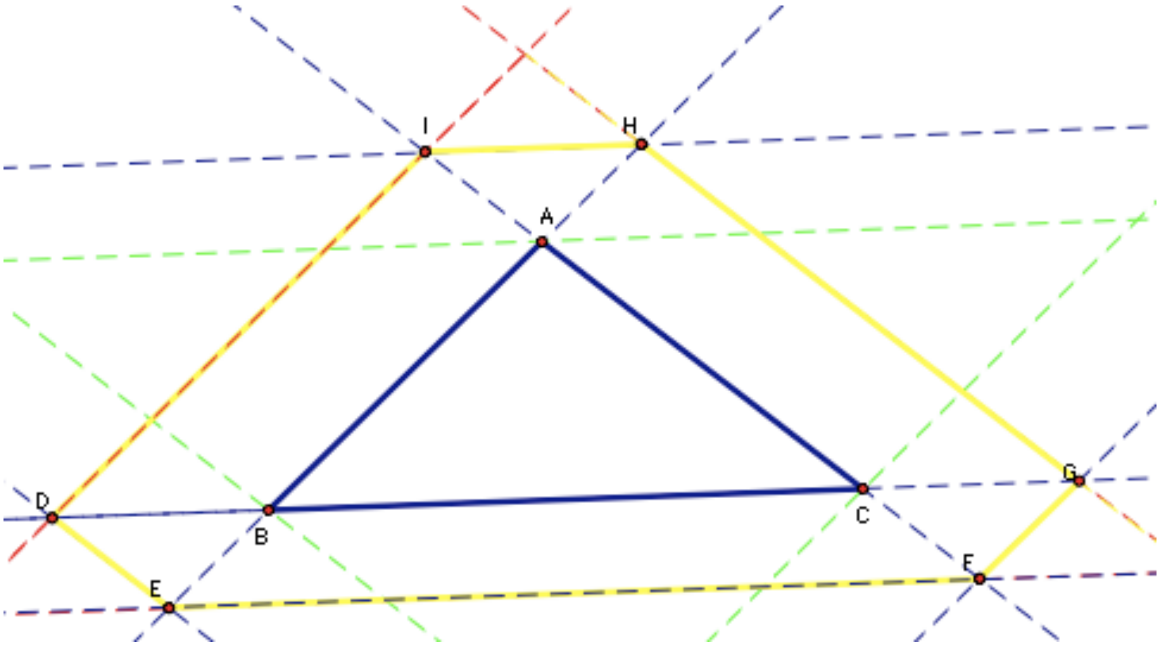


Nothing special can be noted that is different from starting at any arbitrary point.

3. If Barney starts on a point outside of ΔABC

If Barney starts at a point outside of ΔABC, then when he reaches a 'wall' that is an extension of a side of the triangle ABC then he does not reflect from the wall but continues 'through' it, slightly changing direction to follow a parallel path to the other sides. He thus covers a distance that is grater than the perimeter of triangle ABC. This distance is equal to the perimeter of ΔABC plus twice the perimeter of three small triangles formed at the points when Barney changes direction. This is also the same distance of the perimeter of ΔABC plus twice the sum of the three short segments parallel to sides AB, BC and AC.

Proof



Using opposite sides of parallelograms are congruent

Let DE = IA= CF = x

Let GF = HA= EB = y

Let IH = DB = CG = z

Perimeter of ΔABC = AB + BC + AC

Distance that Barney travels is x + EF + y + GH + z+ ID, where EF = BC + z, GH = AC + x, and ID = AB + y, by opposite sides of parallelograms are congruent.

Distance traveled = x + BC + z + y + AC + x + z+ AB + y.

= AB + BC + AC + 2x + 2y + 2z

It can also be stated that if directed segments are uses then given a change in direction (sections where he travels in opposite direction from the direction inside) has the opposite sign then the distance travelled by Barney would be equal to the perimeter of the ΔABC.

Thus Distance traveled = (-x) + BC + z + (-y) + AC + x + (- z)+ AB + y = AB + BC + AC = Perimeter of ΔABC.
