The University of Georgia

Mathematics Education Program

J. Wilson, EMAT 6600

Cone Half Full

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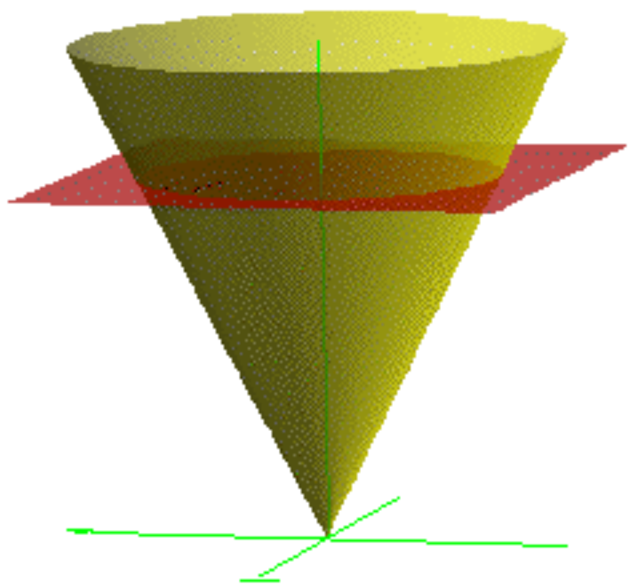
Goal: To find the volume of a half full cone

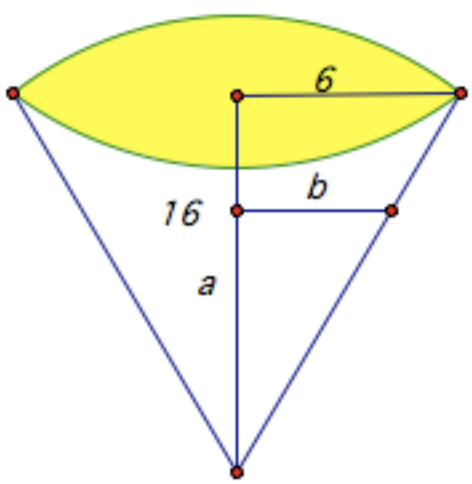
Problem

Consider a right circular cone with diameter of 12 cm for the base and altitude of 16 cm. If the cone is held with its vertex down and water placed into it until it is half full, what is the depth of the water?



Solution

The diagram below shows the sections that the cone at the full and half full phases.



Full volume and solve for the height and radius of a cone with half the volume

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi 6^2(16)$

$= 192\pi$

Now using similar triangles, the radius (b) of the less than full cone can be derived as follows:

$\frac{16}{6} = \frac{a}{b}$

So $b = \frac{3}{8}a$

and the depth (height) of the cone half full

$a = \frac{8}{3}b$

So the radius at half volume $b = \frac{3}{8}a$

Substituting the value of the radius into to the volume of the cone to find the height

$V = \frac{1}{3}\pi r^2 h$

$192\pi = \frac{1}{3}\pi(\frac{3}{8}h)^2 h$

So the volume of the half full cone is: $96\pi = \frac{1}{3}\pi(\frac{3}{8}a)^2 a$

$a^3 = 2048$

$a = 8\sqrt[3]{4}$

The radius of the half full cone

$b = \frac{3}{8}a = b = \frac{3}{8}(8\sqrt[3]{4}) = 3\sqrt[3]{4}$

The depth of water in the cone half full is $8\sqrt[3]{4}$ cm

And the radius is $3\sqrt[3]{4}$ cm.

Find the depth *a* when a cone of radius *r* and height *h* is half full of water? Use this general solution to find the solution for this particular cone.

Let h and r represent the radius and height of the full cone then the and let a and b represent the height and radius of the less than full cone then the following proportion holds:

$\frac{h}{r} = \frac{a}{b}$

If we introduce a constant ratio $\frac{x}{y}$, where x and y are non zero integers, as the relationship between h and r such that

$h = \frac{x}{y}r$, then similarly $a = \frac{x}{y}b$.

The volume of the full cone $V = \frac{1}{3}\pi r^2 h$ then for the half cone $\frac{1}{2} V = \frac{1}{3}\pi b^2 a$

$\Rightarrow \frac{1}{2}(\frac{1}{3}\pi r^2 h) = \frac{1}{3}\pi b^2 a$

$\Rightarrow \frac{1}{2} r^2 h = b^2 a$

Substituting $h = \frac{x}{y} r$ and $a = \frac{x}{y} b$

$\Rightarrow \frac{1}{2} r^2 \frac{x}{y} r = b^2 \frac{x}{y} b$

$\Rightarrow \frac{1}{2} r^3 = b^3$

$\Rightarrow \quad b = \sqrt[3]{(1/2)r\text{^}3}$

so $a = \frac{x}{y} \sqrt[3]{(1/2)r\text{^}3}$

Recall from the previous example that the h = 16 and r = 6 so the constant ratio was 8/3

Plugging this into the above equation gives:

$a = \frac{8}{3} \sqrt[3]{(1/2)(6)\text{^}3}$

$a = 8\sqrt[3]{4}$

and

$b = \frac{3}{8} a = b = \frac{3}{8}(8\sqrt[3]{4}) = 3\sqrt[3]{4}$

Let k be the fraction of cone that is filled and a the depth of water, we know that when k = ½ the height of water in the cone is $8\sqrt[3]{4}$. We develop a graph for the range of k from 0 to 1 and graph it.

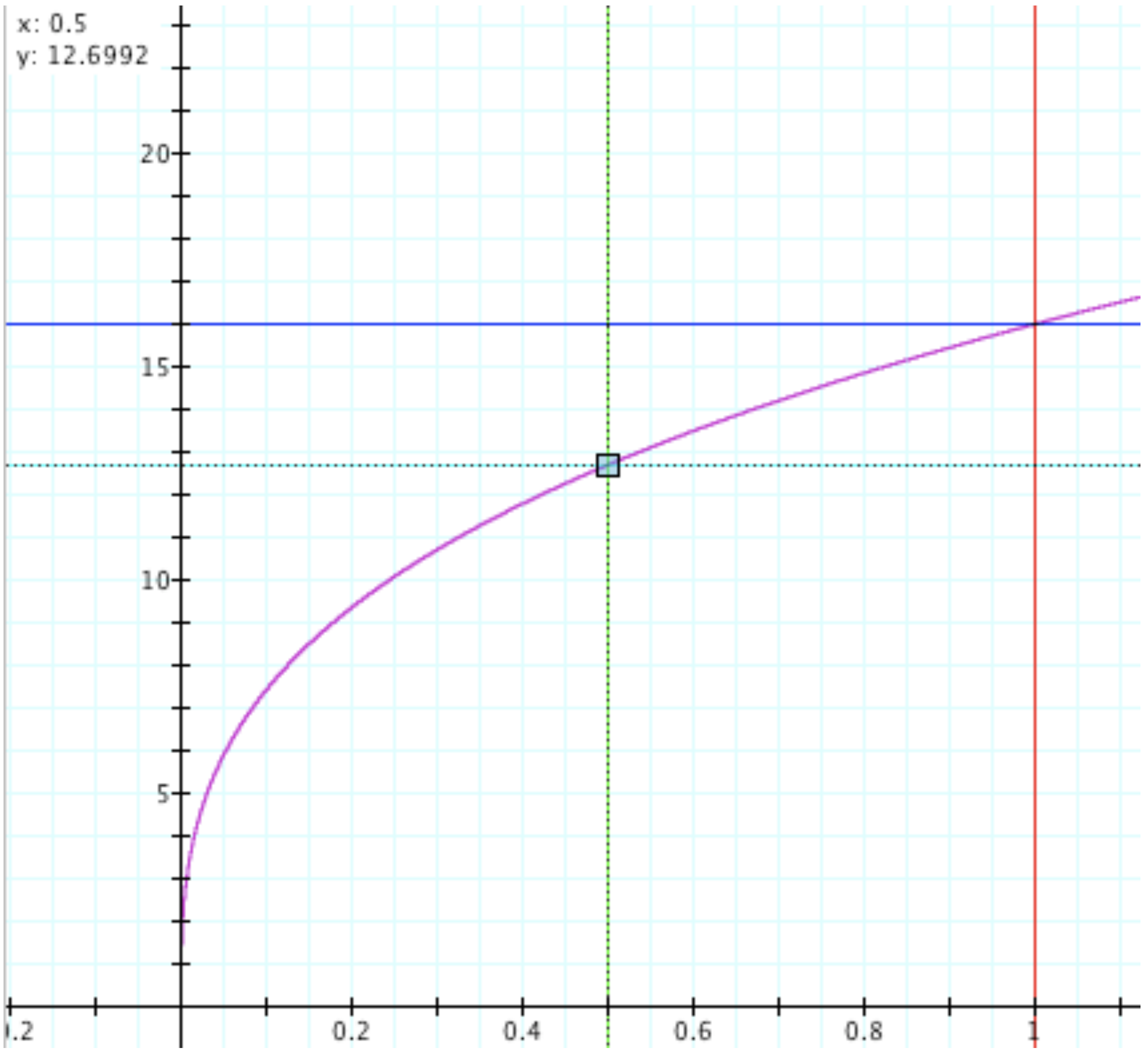
Recall

$a = \frac{8}{3} \sqrt[3]{(1/2)(6)\text{^}3}$

let the ½ be k then

$a = \frac{8}{3}\sqrt[3]{k(6)^3}$

$a = 16\sqrt[3]{k}$



The values of *a* range lie along the vertical axis. The values of *k* lie on the horizontal axis. The pink curve shows the function $a = 16\sqrt[3]{k}$. The point plotted by the blue box represents the point at which the cone is half full.