Goal: to find the area of a field a goat can graze that

**Problem**

Given a square building with each side 20 ft. in a field. A goat is tethered to a point on the base of the building. If the tether is 40 ft. long, what is the area over which the goat can graze?

What if the tether is 50 ft. long?

Does it matter where the tether is tied to the side of the barn? That is, will the area be the same regardless where the tether is connected? Why?
**Standard form solution**

When the goat moves along its 40 foot rope tethered to the barn, it can sweep out a half circle of radius 40 up the straight line formed with the barn:

Let the radius be $r_0 = 40$

Then $A_0 = \frac{1}{2} \pi (40)^2$
Let $x$ be the distance from the corner of the barn that the goat is tethered. Then the max length of the rope to the left is $40 - x$.

Let $r_1$ be $40 - x$

Then $A_1 = \frac{1}{4} \pi (40 - x)^2 = \frac{1}{4} \pi (1600 - 80x + x^2)$

After goat pivots at the top left the max distance is $20 - x$.

Let $r_2$ be $20 - x$

Then $A_2 = \frac{1}{4} \pi (20 - x)^2 = \frac{1}{4} \pi (400 - 40x + x^2)$

Then the max length of the rope to the right is $20 + x$.

Let $r_3$ be $20 + x$

Then $A_3 = \frac{1}{4} \pi (20 + x)^2$

$= \frac{1}{4} \pi (400 + 40x + x^2)$

After goat pivots at the top right the max distance is $x$.

Let $r_4$ be $x$

Then $A_4 = \frac{1}{4} \pi (x)^2$

Finally when the total area grazed by the goat is the sum of the areas
\[ A = A_0 + A_1 + A_2 + A_3 + A_4 \]
\[ = \frac{1}{2} \pi (40)^2 + \frac{1}{4} \pi (1600 - 80x + x^2) + \frac{1}{4} \pi (400 - 40x + x^2) + \frac{1}{4} \pi (400 + 40x + x^2) \]
\[ + \frac{1}{4} \pi (x)^2 \]
\[ = \frac{1}{4} \pi (3200 + 1600 - 80x + x^2 + 400 - 40x + x^2 + 400 + 40x + x^2 + x^2) \]
\[ = \frac{1}{4} \pi (5600 - 80x + 4x^2) \]
\[ = \pi (x^2 - 20x + 1400) \]

**Vertex Form Solution**

Use the two end points and midpoint of the side of the barn to which the goat is tethered to locate 3 point on the parabola formed.

Using point 0, 10, and 40 and area formula \( A = \pi r^2 \) goat tethered at 10

<table>
<thead>
<tr>
<th>Possible lengths</th>
<th>Area grazed</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>1600</td>
</tr>
<tr>
<td>30</td>
<td>900</td>
</tr>
</tbody>
</table>
So we have three points on the parabola the minimum (10, 1300\pi) and the two endpoints (0, 1400\pi) and (40, 1400\pi).

Recall the vertex form of a quadratic equation is
\[
f(x) = a(x-h)^2 + k
\]

Substituting the values (0, 1400\pi) and (10, 1300\pi) into the equation gives:
\[
1400\pi = a(0-10)^2 + 1300\pi
\]
\[
100\pi = 100a
\]
\[
a = \pi
\]
So the equation for the area formula is
\[ f(x) = \pi (x-10)^2 + 1300\pi \]

If the rope is 50 ft long there will be some overlap of the areas to the top of the barn.

It does not matter what side of the barn the goat is tied to begin with. As it moves around it will eventually graze out all that is within a given radius.