The University of Georgia

Mathematics Education Program

J. Wilson, EMAT 6600

Inscribed Circles

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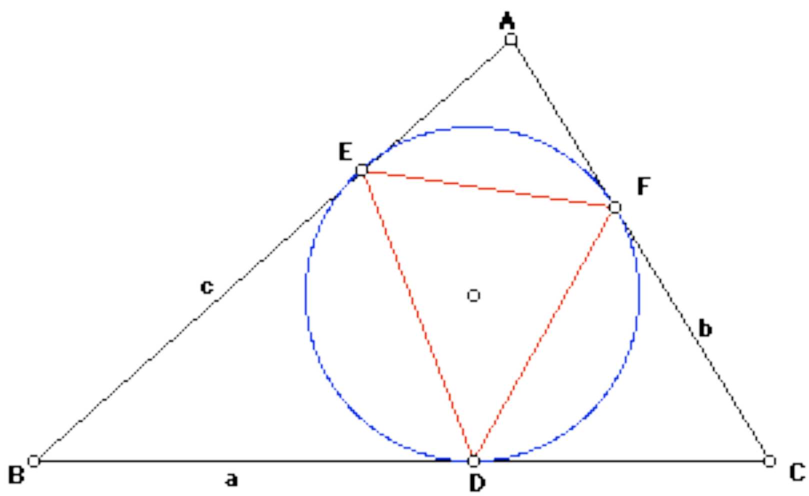
Goal: To investigate angles, triangles and concurrency in incircles

Problem

Given triangle ABC with side lengths a, b, and c. Let D, E, and F be the points of tangency of the incircle, as shown.



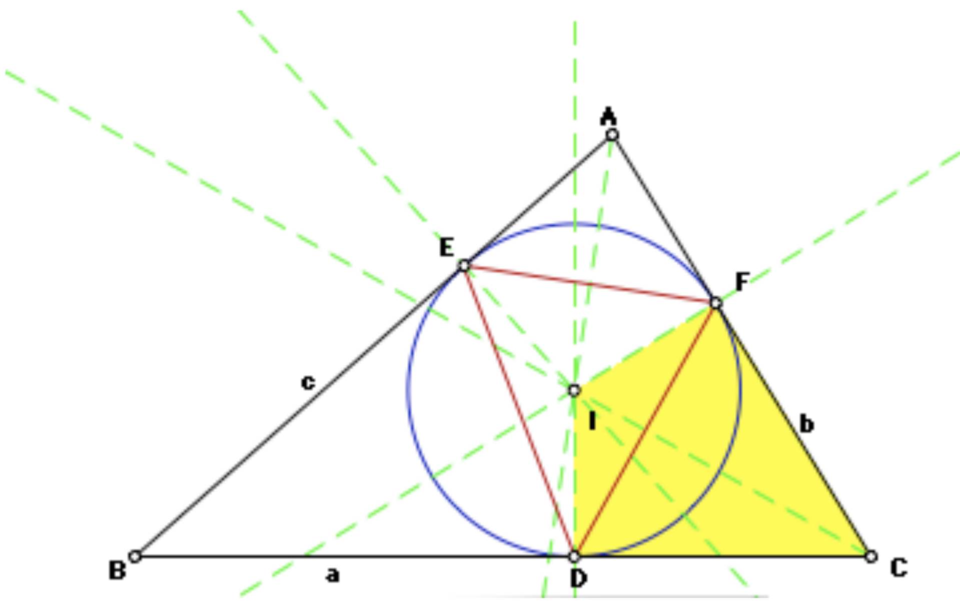
(a) Prove that triangle DEF is acute, that is, that the triangle determined by the points of tangency of the incircle is always acute.

Solution

Using the theorem from central and peripheral angles, we state that: when subtended for the same chord, the angle at the center is twice the angle at the periphery on a circle. In other words the central angle is twice the peripheral angle, from the same chord. Another name for peripheral angle is inscribed angle. So the theorem is sometimes stated as the central angle is twice the inscribed angle.

Let the point at the center be I.

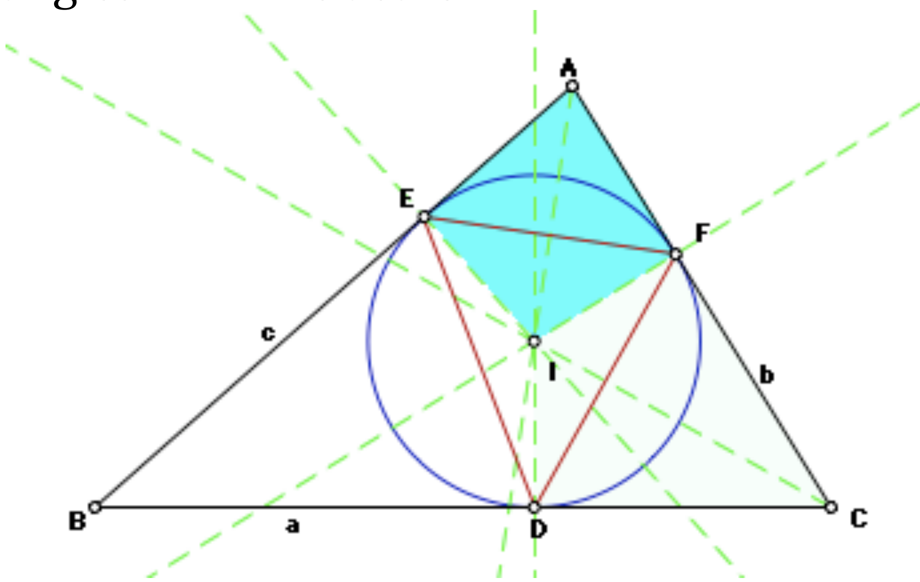
In the following diagram angles DEF, FDE and EFD are inscribed angles whereas angles DIF, FIE and DIE are central angles.



Consider CDIF is a quadrilateral, CD is tangent to circle so <CDI is right angle,

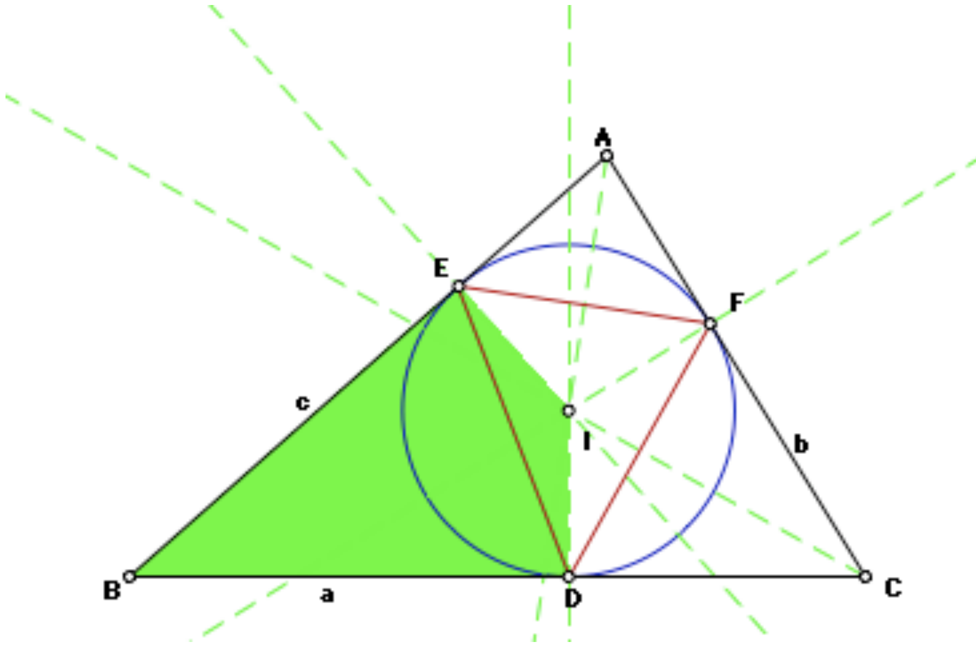
CF is tangent to circle so <CFI is right angle. Hence <DIF and <DCF are supplementary. Thus <DIF is less

than 180^o. If <DIF is less than 180^o then <DEF is less than 90^o by the theorem of central and inscribed angles. <DEF is acute.



Consider AEIF is a quadrilateral, AE is tangent to circle so <AEI is right angle,

AF is tangent to circle so <AFI is right angle. Hence <EIF and <EAF are supplementary. Thus <EIF is less than 180^o. If <EIF is less than 180^o then <EDF is less than 90^o by the theorem of central and inscribed angles. <EDF is acute.



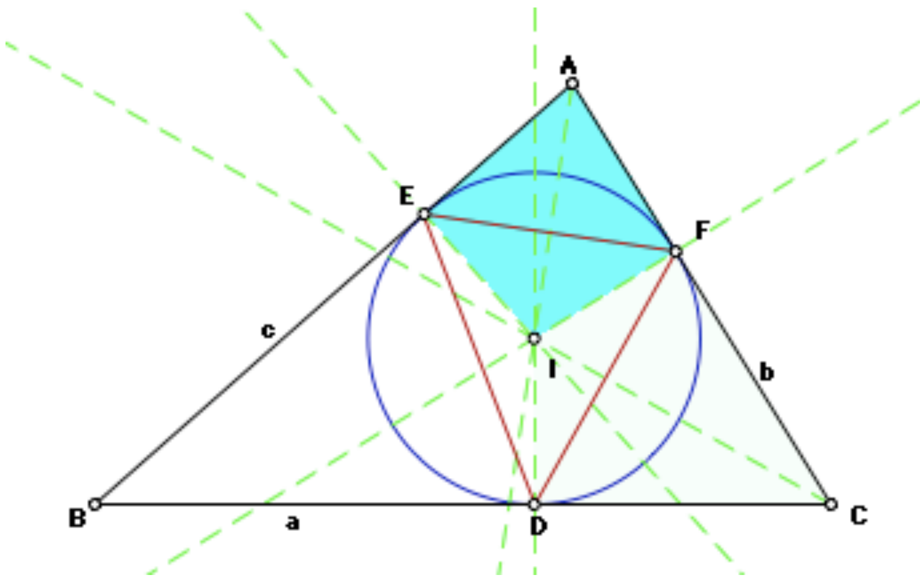
Consider BDIE is a quadrilateral, BE is tangent to circle so <BEI is right angle,

BD is tangent to circle so <BDI is right angle. Hence <DIE and <DBE are supplementary. Thus <DIE is less than 180^o. If <DIE is less than 180^o then <DFE is less than 90^o by the theorem of central and inscribed angles. <DFE is acute.

Now because <DFE, <EDF, <DEF are all acute then ΔDEF is an acute triangle.

(b) Find the area of triangle DEF in terms of a, b, and c.

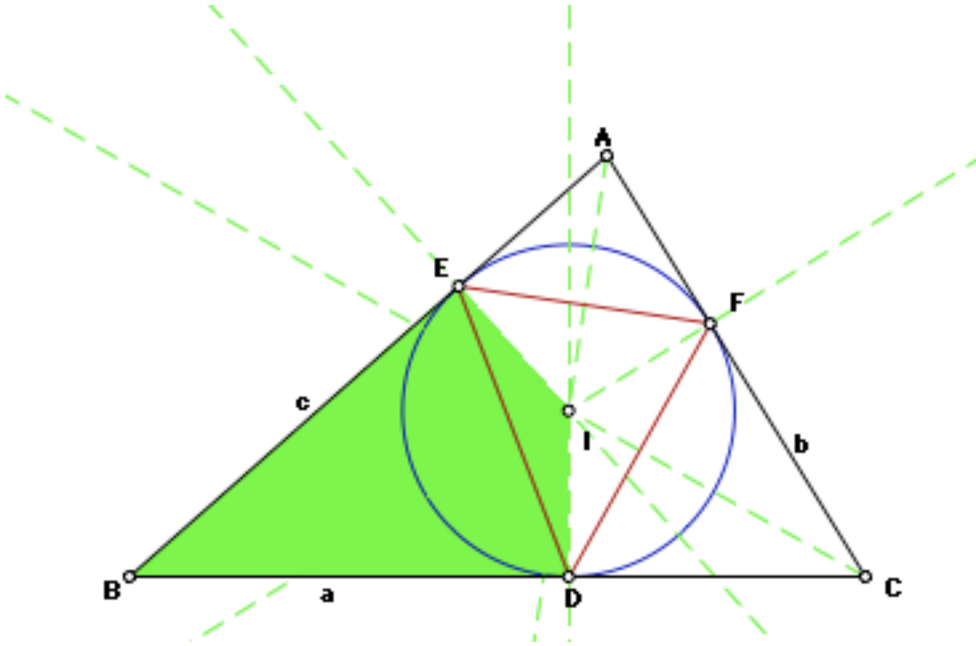
Solution



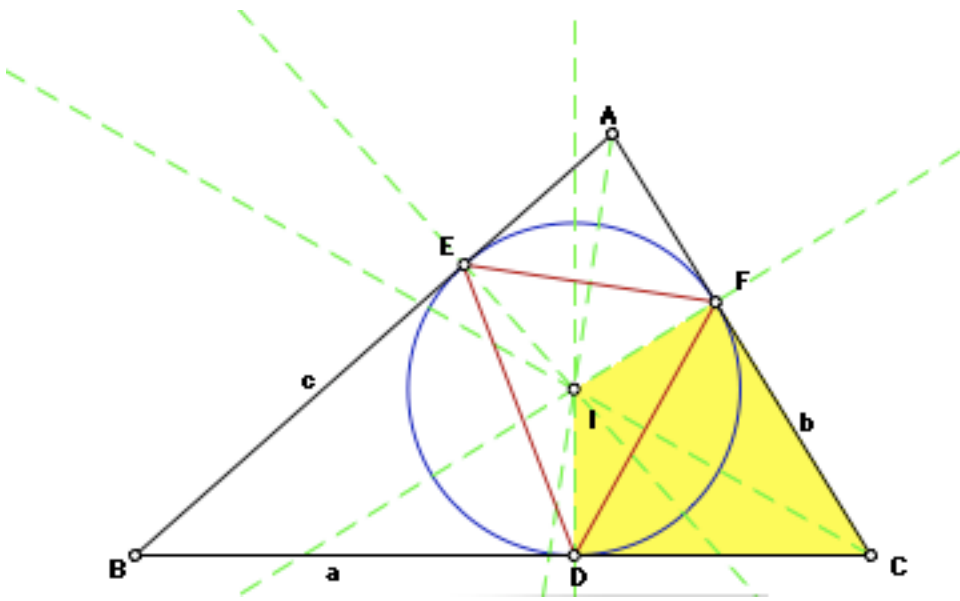
Again, consider quadrilateral AEIF. m<AEI = m< AFI = 900. <EAF and <EIF are supplementary. Thus sin<EAF = sin<EIF or both equal sinA. EI = FI = r, radii of inscribed circle.

Area of Δ EIF = $\frac{1}{2}(EI)(FI)\sin < EIF$ = $\frac{1}{2}$r²sinA

In an analogously manner, consider quadrilaterals BDIE and CDIF. In a similar manner the areas of ΔDIE and ΔDIF can be formulated as:



Area of Δ EID = $\frac{1}{2}(EI)(ID)\sin < EID = \frac{1}{2}r^2\sin B$



Area of Δ DIF = $\frac{1}{2}(DI)(FI)\sin < DIF = \frac{1}{2}r^2\sin C$

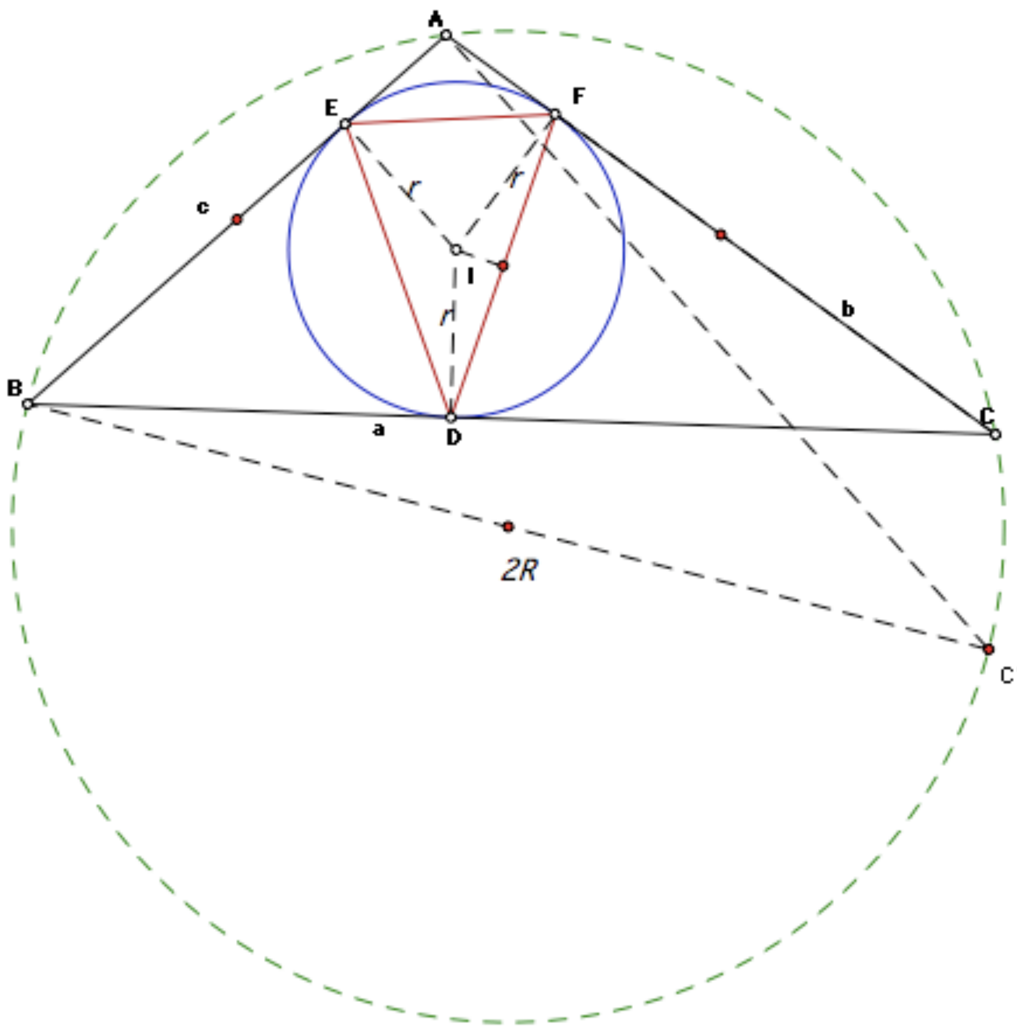
Thus

Area of Δ DEF = Area of Δ EIF + Area of Δ EID + Area of Δ DIF

$= \frac{1}{2}r^2\sin A + \frac{1}{2}r^2\sin B + \frac{1}{2}r^2\sin C \quad = \frac{1}{2}r^2(\sin A + \sin B + \sin C)$

But we want to express the area in terms of the sides a, b, c

Recall the sine rule which states that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the radius of the circumscribed circle.



Consider the circle circumscribing ΔABC, and C' is subtended from chord c, then <C' = <C. So, sinC' = c/2R = sin C.

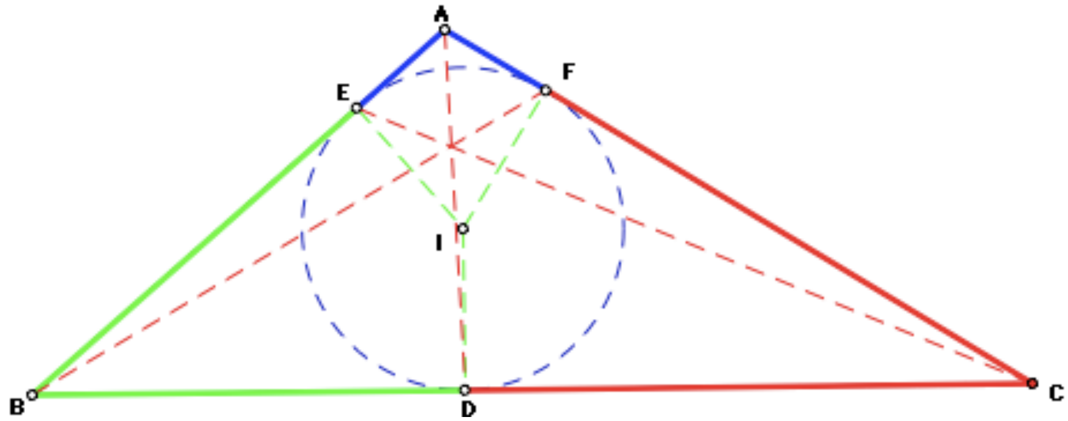
In a similar manner sinA= a/2R and sinB = b/2R.

Now plugging these into Area Δ DEF = $\frac{1}{2}r^2(\sin A + \sin B + \sin C)$ gives

Area Δ DEF = $\frac{1}{2}r^2\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right) = \frac{1}{2}r^2\left(\frac{a+b+c}{2R}\right) = \frac{1}{4}r^2\left(\frac{a+b+c}{R}\right)$

(c) Show that AD, BF, and CE are concurrent.

Solution



Using Ceva's theorem the angle bisectors of triangle ABC are concurrent if $\frac{DB}{DC} * \frac{FC}{FA} * \frac{EA}{EB} = 1$

Consider ΔBEI and ΔBDI. m<BEI = m<BDI = 90º . m<EBI ≈ m<DBI by angle bisector, thus m<EIB ≈ m<DIB. BI≈ BI by reflexive property. Hence ΔBEI ≈ ΔBDI, by AAS; Particularly, DB≈ EB. [1]

Consider ΔFCI and ΔDCI. m<CDI = m<CFI = 90º . m<FCI ≈ m<DCI by angle bisector, thus m<DIC ≈ m<FIC. CI≈ CI by reflexive property. Hence ΔFCI ≈ΔDCI, by AAS; Particularly, DC≈ FC. [2]

Consider ΔAEI and ΔAFI. m<AEI = m<AFI = 90º . m<EAI ≈ m<FAI by angle bisector, thus m<EIA ≈ m<FIA. AI≈ AI by reflexive property. Hence ΔAEI ≈ ΔAFI, by AAS; Particularly, FA≈ EA. [3]

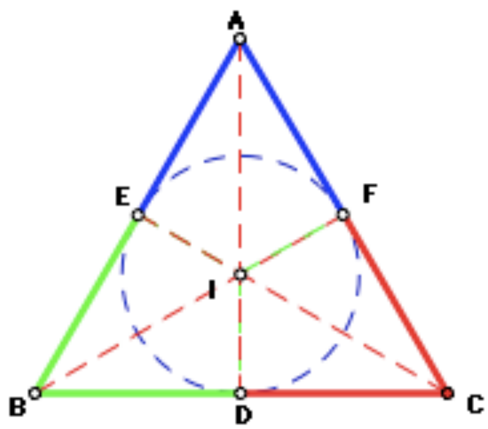
We want to find if $\frac{DB}{DC} * \frac{FC}{FA} * \frac{EA}{EB} = 1$ so we plug in the results in [1],[2], and [3] into it

Thus we have $\frac{DB}{DC} * \frac{FC}{FA} * \frac{EA}{EB} = \frac{DB}{DC} * \frac{DC}{FA} * \frac{FA}{DB} = 1$

Hence the segments AD, BF, and CE are concurrent.

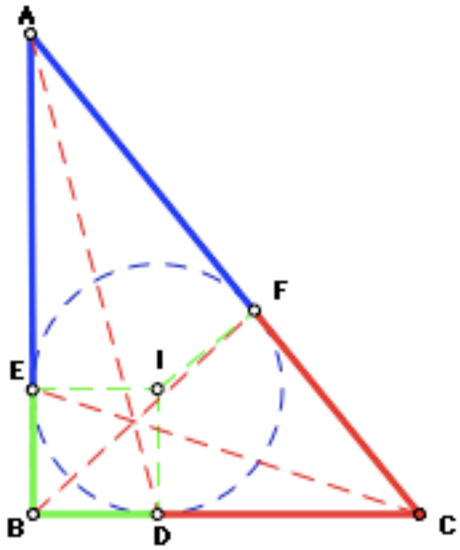
Explore this point of concurrency as the shape of triangle ABC is varied.

Equilateral Triangle



The angle bisectors and the perpendiculars from the center to the sides of the lie along the same path, where AD is on the angle bisector of A, BF is on the bisector of B and CE is on the bisector of C . The perpendiculars and the angle bisectors pass through the midpoints of the sides of triangle ABC. All the segment ratios are equal so the Ceva's theorem holds and the segments AD, BF, and CE are concurrent.

In the right triangle the segments AD, BF, and CE are concurrent.



The concurrency also holds in the obtuse and acute

triangles,