The University of Georgia

Mathematics Education Program

J. Wilson, EMAT 6600

Sublime Triangle

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Goal: To prove that the ratio of the lateral side of a sublime triangle to its base is the golden ratio

Problem

Sublime Triangle

(Also called the **Golden Triangle**)

The ***Sublime Triangle*** is an isosceles triangle with angles of measure 36, 72, and 72 degrees. It is the only triangle with angle measures in the ratio 1:2:2.

Prove that the ratio of a lateral side of a sublime triangle to its base is the golden ratio,

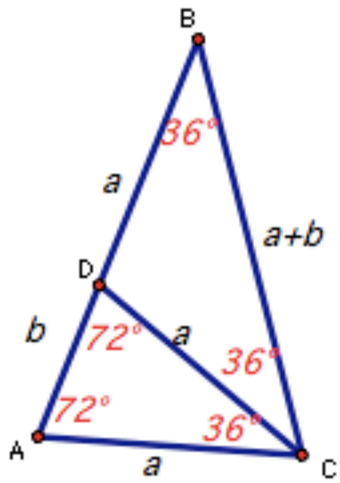
$$\varphi = \frac{1 + \sqrt{5}}{2}$$

Alternatively, if we DEFINE the Sublime triangle as an isosceles triangle with the ratio of the measure of the lateral side to the measure of the base to be the Golden Ratio, then we can prove that the angles have measure of 36 - 72 - 72.

Hint: Construct an angle bisector of one of the base angles and extend it to the opposite side, thus producing a smaller triangle similar to the original.

Solution

Sublime Triangle



ΔABC and ΔACD are isosceles

∠ABC ≈ ∠ACD ; ∠BAC ≈ ∠CAD ⇒ΔABC and ΔACD are similar by AA similarity

Checking proportions

$$\frac{a+b}{a} = \frac{a}{b} = \varphi = \frac{1+\sqrt{5}}{2}$$

Proof

$$\frac{a+b}{a} = \frac{a}{b}$$

$$\Rightarrow \frac{a/b + b/b}{a/b} = \frac{a}{b}$$

$$\Rightarrow \frac{\varphi + 1}{\varphi} = \varphi$$

$$\Rightarrow \varphi^2 = \varphi + 1$$

$\Rightarrow \varphi^2 - \varphi - 1 = 0$

$\Rightarrow \varphi = \frac{1+\sqrt{5}}{2}$

Extending

$\varphi^3 = \varphi^2 * \varphi = (\varphi + 1)\,\varphi = \varphi^2 + \varphi = \varphi + 1 + \varphi = 2\,\varphi + 1$

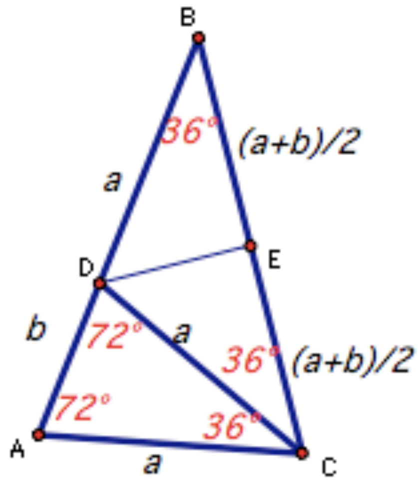
$\varphi^4 = \varphi^3 * \varphi = (2\,\varphi + 1)\,\varphi = 2\varphi^2 + \varphi = 2\varphi + 2 + \varphi = 3\,\varphi + 2$

$\varphi^5 = \varphi^4 * \varphi = (3\,\varphi + 2)\,\varphi = 3\varphi^2 + 2\varphi = 3\varphi + 3 + 2\varphi = 5\,\varphi + 3$

$\varphi^6 = \varphi^5 * \varphi = (5\,\varphi + 3)\,\varphi = 5\varphi^2 + 3\varphi = 5\varphi + 5 + 3\varphi = 8\,\varphi + 5$

$\varphi^7 = \varphi^6 * \varphi = (8\,\varphi + 5)\,\varphi = 8\varphi^2 + 5\varphi = 8\varphi + 8 + 5\varphi = 13\,\varphi + 8$

$\varphi^8 = \varphi^7 * \varphi = (13\,\varphi + 8)\,\varphi = 13\varphi^2 + 8\varphi = 13\varphi + 13 + 8\varphi = 21\,\varphi + 13$



If we drop a perpendicular from D to side BC to intersect at point E, then BE = (a+b)/2, ∠BED is right angle

Cos 36 = BE/BD = ((a+b)/2)/a = $\frac{a+b}{a}/2 = \frac{\varphi}{2}$