The University of Georgia

Mathematics Education Program

J. Wilson, EMAT 6600

Minimum Surface Area of a Can

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Goal: To find the minimum surface area of a 12 oz can.

Problem

In packaging a product in a can the shape of right circular cylinder, various factors such as tradition and supposed customer preferences may enter into decisions about what shape (e.g. short and fat vs. tall and skinny) can might be used for a fixed volume. Note, for example, all 12 oz. soda cans have the same shape -- a height of about 5 inches and a radius of about 1.25 inches. Why?

What if the producer's decision was based on minimizing the material used to make the can? This would mean that for a fixed volume **V** the shape of the can (e.g. the radius and the height) would be determined by the minimum surface area for the can. What is the relationship between the radius and the height in order to minimize the surface area for a fixed volume?

Solution

The soda can probably have the same dimensions height of about 5 inches and radius of 1.25 inches because these are the most economical dimensions for the producer. In other words these are the dimensions that allow the can to be produced at minimum cost.

Let us assume that the material for making the can is uniformly distributed and given that the volume is fixed we can apply the volume of the can solve for h pug this into the surface area formula and then use the AM – GM rule to find the minimum surface area.

Recall the formula for volume is $V = \pi r^2 h$

Then $h = \frac{V}{\pi r^2}$ [1]

Now the surface area of the can is given by $SA = 2\pi r^2 + 2\pi rh$ [2]

So plugging [1] into [2] gives:

$$SA = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2}$$

Now we want to use the AM – GM rule so we simplify the expression to the left and break it into three parts

$$SA = 2\pi r^2 + \frac{V}{r} + \frac{V}{r}$$

Applying the AM-GM

$2\pi r^2 + \frac{V}{r} + \frac{V}{r} \geq 3\sqrt[3]{2\pi r^2 * \frac{V}{r} * \frac{V}{r}} = 3\sqrt[3]{2\pi V^2}$

To minimize SA we make it equal to the constant so

$2\pi r^2 = \frac{V}{r}$

$2\pi r^3 = V$

$2\pi r^3 = \pi r^2 h$

$2r = h$

Thus the minimum surface area of the 12 oz can is achieve when the height is twice the radius of the can.

Solution Using Calculus

Set up a primary equation from $SA = 2\pi r^2 + 2\pi rh$

Using secondary equation $V = \pi r^2 h$ solve for h, $h = \frac{V}{\pi r^2}$,

and plug into the primary equation. The primary equation with one independent variable and one dependent variable is

$SA = 2\pi r^2 + \frac{2V}{r}$ (treating V as fixed or constant).

Differentiate to find the minimum

$dA/dr = 4\pi r - 2V/r^2$

To show it is really a minimum take the second derivative

$d^2A/dr^2 = 4\pi + 4V/r^3$

Since d^2A/dr^2 is positive for all positive r, the critical point must be a minimum.

The constraints are that r and h must both be positive.

Set first derivative equal to 0 and solve for r

$4\pi r - 2V/r^2 = 0 \Rightarrow r = \sqrt[3]{\frac{V}{2\pi}}$

Plugging in $V = \pi r^2 h$ into the above gives

$r = \sqrt[3]{\frac{\pi r\hat{}2h}{2\pi}} = \sqrt[3]{\frac{r\hat{}2h}{2}}$

$r^3 = (r^2h)/2$

$2r = h$

So the minimum surface area is when the height is about twice the radius.

Going back to the initial question, we saw that the radius is 1.25 inches and the height is 5. Given this we know that the distribution is not uniform because if it was then the height would be exactly equal to twice the radius. Other factors have to be taken into consideration in making the soda cans such as how much material should go into making the seams.