



The University of Georgia

Mathematics Education Program

J. Wilson, EMAT 6600

Sum and Product

By Leighton McIntyre

Goal: to prove that if the sum of n positive numbers is 1 then the sum is greater than or equal to 1

Problem

The problem examines the theorem that if the product of n positive real numbers is 1, then the sum is greater than or equal to n .

1. Let a and b be two positive numbers such that $ab = 1$.

Prove that $a + b \geq 2$

2. Let r , s , and t be three positive real numbers such that $rst = 1$.

Prove that $r + s + t \geq 3$

3. Let $a_1, a_2, a_3, \dots, a_n$ be n positive real numbers such that

$$a_1 a_2 a_3 \dots a_n = 1$$

Prove that $a_1 + a_2 + a_3 + \dots + a_n \geq n$

Clearly, the proof of 3 implies the first two.

Looking at the first as a separate problem, however, 3 different ways to approach the Proof:

Given $ab = 1$.

$$\text{then } \sqrt{ab} = 1 \text{ and } 2\sqrt{ab} = 2$$

$$\text{But } a + b \geq 2\sqrt{ab} \text{ by AM-GM}$$

$$\text{and so } a + b \geq 2$$

$$\text{or } \dots \quad ab = 1 \Rightarrow b = \frac{1}{a}$$

$$a + b = a + \frac{1}{a} \geq 2\sqrt{a\left(\frac{1}{a}\right)} \text{ by AM-GM}$$

$$\text{so } a + b \geq 2$$

Alternatively,

$$a + b \geq 2\sqrt{ab} \text{ by AM-GM}$$

and since $ab = 1$ is given

$$a + b \geq 2$$

HINT: Use the AM-GM Inequality to prove problems 2 and 3.

$$2) \text{ Given } rst = 1 \text{ then } \sqrt[3]{rst} = 1 \text{ and } 3\sqrt[3]{rst} = 3$$

$$\text{Then } r + s + t \geq 3$$

Alternatively,

$$r + s + t \geq 3\sqrt[3]{rst} \text{ by AM-GM}$$

Since $rst = 1$ as given then

$$r + s + t \geq 3$$

$$3) \text{ Given } a_1 a_2 a_3 \dots a_n = 1 \text{ then } \sqrt[n]{a_1 a_2 a_3 \dots a_n} = 1 \text{ and}$$

$$n\sqrt[n]{a_1 a_2 a_3 \dots a_n} = n$$

$$\text{Then } a_1 + a_2 + a_3 + \dots + a_n \geq n$$

Alternatively,

$a_1 + a_2 + a_3 + \dots + a_n \geq n \sqrt[n]{a_1 a_2 a_3 \dots a_n}$ by AM-GM

Since $a_1 a_2 a_3 \dots a_n = 1$ as given then

$a_1 + a_2 + a_3 + \dots + a_n \geq n$

Discussion

Even though these problems were relatively easy to calculate, they have very important uses particularly in the field of Statistics.
