Maximum Area of a Triangle

**Problem:** Use the Arithmetic Mean -- Geometric Mean Inequality to show that the maximum area of a triangular region with a given perimeter is attained when the triangle is equilateral.

**Solution:**

Semi-perimeter of the triangle, \( S = \frac{a + b + c}{2} \Rightarrow 2S = a + b + c \)

We can find the area using Heron’s Formula, \( A = \sqrt{s(s - a)(s - b)(s - c)} \)

Using AM-GM Inequality,

\[
(s - a)(s - b)(s - c) \leq \left( \frac{(s - a) + (s - b) + (s - c)}{3} \right)^3
\]

\[
= \left[ \frac{3s - (a + b + c)}{3} \right]^3
\]

Since, \( 2S = a + b + c \), we have

\[
(s - a)(s - b)(s - c) \leq \left[ \frac{3s - 2s}{3} \right]^3 = \left[ \frac{s}{3} \right]^3 = \frac{s^3}{27}
\]

\( (s - a)(s - b)(s - c) \leq \frac{s^3}{27} \) with equality when \( s - a = s - b = s - c \)

Since \( a + b + c \) is a constant, then \( S = \frac{a + b + c}{2} \) is also a constant

Hence \( a = b = c \) \( \Rightarrow \) An Equilateral Triangle.

So, \( A = s \left( \frac{s}{3} \right)^3 = \frac{s^4}{27} \).