Conical Frustum

A Conical Frustum is a Frustum created by slicing the top off a cone (with the cut made parallel to the base), forming a lower base and an upper base that are circular and parallel. Let $h$ be the height, $R$ the radius of the lower base, and $r$ the radius of the upper base as pictured below:

The Volume of Frustum:

The Volume of the Frustum could be found using the formula:

$$ V = \frac{\pi h}{3} (R^2 + Rr + r^2) $$

Now, let’s derive the formula without using calculus.

Consider the cone before it was cut. Let the height of the cut be $H$.

The volume of the cone, $V = \frac{1}{3} \pi r^2 h$

The volume of the original pre-cut cone, $V_o = \frac{1}{3} \pi R^2 (H + h)$

The volume of the cut part, $V_c = \frac{1}{3} \pi r^2 H$

Now, to get the volume of the frustum ($V_F$), we have to subtract the volume of the cut part from the volume of the original cone. So,

$$ V_F = V_o - V_c $$

$$ = \frac{1}{3} \pi R^2 (H + h) - \frac{1}{3} \pi r^2 H $$

$$ = \frac{1}{3} \pi R^2 H - \frac{1}{3} \pi R^2 h - \frac{1}{3} \pi r^2 H $$
Now, consider the original pre-cut cone. The triangles formed by the height and the bases are similar by AA similarity and the sides are proportional.

Hence, \( \frac{H}{r} = \frac{H+h}{R} \) \( \ldots \ldots \) (ii)

So we have,

\[ \begin{align*}
HR & = r(H + h) \\
\Rightarrow HR & = Hr + hr \\
\Rightarrow HR - Hr & = hr \\
\Rightarrow H(R - r) & = hr
\end{align*} \]

Substitute \( H(R - r) \) in equation (i), we have

\[ V_F = \frac{1}{3} \pi [R^2 h + hr(R + r)] \\
= \frac{1}{3} \pi h[R^2 + r(R + r)] \\
= \frac{1}{3} \pi h[R^2 + Rr + r^2]
\]

Hence the formula of volume of the Frustum.

**Total Surface Area of a Frustum:**

Now, let’s examine how to find the Surface Area of the frustum.

We know the lateral area of a right circular cone is \( \pi rs \). For the right circular cone here, let \( L \) be the *slant height* and \( r \) and \( R \) be the top and bottom radii. Then,

\[ L = \sqrt{(H + h)^2 + R^2} \]
So, the lateral area of the pre-cut cone, \( LA_o = \pi r \sqrt{(H + h)^2 + R^2} \)

The lateral area of the cut part, \( LA_c = \pi r \sqrt{H^2 + r^2} \)

Hence, area of lateral fractum, \( LA_f = LA_o - LA_c = \pi r \sqrt{(H + h)^2 + R^2} - \pi r \sqrt{H^2 + r^2} \)

To find the total surface area of the frustum, we need to add the area of the base. Additionally, the area of the base is the area of circle with radius \( r \).

Hence, the total surface area of a frustum, \( SA_f = \pi r \sqrt{(H + h)^2 + R^2} - \pi r \sqrt{H^2 + r^2} + \pi r^2 \).

**Lateral Area of a Frustum of a Right Circular Cone:**

The lateral area of a frustum of a right circular cone is given by

\[ A = \pi (R + r) L \]

Where, \( R = \) radius of the lower base, \( r = \) radius of the upper base, and \( L = \) the length of the lateral side.

Now, the lateral area of right circular cone is the difference of the areas of sections of a circle of radii \( L_1 \) and \( L_2 \), and common central angle \( \theta \) (as pictured below):

![Diagram](image)

By ratio and proportion:

\[ \frac{L_1}{R} = \frac{L}{R - r} \]

\[ L_1 = \frac{RL}{R - r} \]
And, from the figures:

\[ L_2 = L_1 - L \]

\[ = \frac{RL}{R-r} - L \]

\[ = \frac{RL - (R-r)L}{R-r} \]

\[ = \frac{rL}{R-r} \]

The length of arc is the circumference of the base:

\[ s_1 = 2\pi R \]

\[ s_2 = 2\pi r \]

Again, from the figure:

\[ A = \frac{1}{2} s_1 L_1 - \frac{1}{2} s_2 L_2 \]

\[ = \frac{1}{2} (2\pi R) \left( \frac{RL}{R-r} \right) - \frac{1}{2} (2\pi r) \left( \frac{rL}{R-r} \right) \]

\[ = \frac{\pi R^2 L}{R-r} - \frac{\pi r^2 L}{R-r} \]
Hence,

\[ A = \pi(R + r)L \]

**Volume of a truncated Pyramid with a square base:**

Let, \( A_1 \) = the area of the lower base

\( A_2 \) = the area of the upper base

\( h \) = perpendicular distance between \( A_1 \) and \( A_2 \) (also known as the attitude of frustum)

Note here that \( A_1 \) and \( A_2 \) are parallel to each other.

Now, Volume of frustum, \( V_1 = \frac{1}{3} A_1 y \)

And top of cone, \( V_2 = \frac{1}{3} A_2(y - h) \)

So, \( V = V_1 - V_2 \)

\[ = \frac{1}{3} A_1 y - \frac{1}{3} A_2(y - h) \]
By similar solids rule,

$$\frac{A_2}{A_1} = \left( \frac{y-h}{y} \right)^2$$

$$\sqrt{\frac{A_2}{A_1}} = 1 - \frac{h}{y}$$

$$\frac{h}{y} = 1 - \sqrt{\frac{A_2}{A_1}}$$

$$= \frac{\sqrt{A_1} - \sqrt{A_2}}{\sqrt{A_1}}$$

So,

$$\frac{y}{h} = \frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}}$$

$$y = h \frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}}$$

$$= h \frac{\sqrt{A_1}}{\sqrt{A_1} - \sqrt{A_2}} \cdot \left( \frac{\sqrt{A_1} + \sqrt{A_2}}{\sqrt{A_1} + \sqrt{A_2}} \right)$$

$$y = \frac{A_1 + \sqrt{A_1 A_2}}{A_1 - A_2} h$$

Now, substituting $y$ in equation $(i)$,

$$V = \frac{1}{3} \left( (A_1 - A_2) \left( \frac{A_1 + \sqrt{A_1 A_2}}{A_1 - A_2} h \right) + A_2 h \right)$$

$$V = \frac{1}{3} \left( (A_1 + \sqrt{A_1 A_2}) h + A_2 h \right)$$
\[ V = \frac{h}{3} \left( A_1 + A_2 + \sqrt{A_1 A_2} \right) \]