The University of Georgia

Mathematics Education
EMAT 4680/6680 Mathematics with Technology
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Relations

by

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Growing up we were all learned to count you using the decimal system. Most of us probably gave little thought to how the decimal system was chosen or developed. We have ten fingers, so of course the natural way to count is by tens, what other way could there be! What if we counted by the dozen? By sevens? By twos? What would be better, worse… simply different (or the same) about using number systems with a different base than ten?

Well first… let's just start with not deviating too far from our comfort zone. Nine is pretty darn close to ten isn't it? What happens if we had a 'base nine' system for counting? We call this a nonary system. Well, it starts off 'naturally' enough. One, two, three, four, five, six, seven, eight… Now here's the big question… What comes next? In the decimal system, our next value would be nine, but let's think how our nine might be different in this new nonary system. Don't worry about me, I'll wait…

That's right what's different is that we're at the 'end of the line' so to speak. At least for our single digits! Back in our decimal using days, we used the symbol '10' when this happened. But let me ask you, what is new and different about the number '10' and its representation? It's actually two symbols isn't it? It's the symbol for one, followed by the symbol for none! Now tell me, how would you explain this to someone who hadn't seen your mysterious decimal system before? Why do the symbols '10' mean ten?... I bet you just came up with a satisfactory answer. Well, aren't you a whiz at this? Now, can you tell me what symbol nine could have in our nonary system, and why this makes sense? I'm guessing that some you said that 'nine' would be represented with a '10' in the nonary system, and some of you said '9'. Whoa, using '10' to represent nine! Why does that automatically seem so strange, regardless of what symbols you picked? Well, you ask yourself that, you'll probably find two reasons. First, you are just used to 10 representing ten. The other reasons probably had to do with working in a decimal system, which we no longer are.

So, we have two different propositions. The nonary number line as 1, 2, 3, 4, 5, 6, 7, 8, 10. Or we can write up our number line as 1, 2, 3, 4, 5, 6, 7, 8, 9. Either way you cut it, we have reached or exceeded the point of single digit numbers. In our nonary system nine is significant.

So, what comes after nine in our nonary system? In other words… what is taking the place for 'ten' for representing the number of fingers on our hands? Well, if nine is represented as 10, then what would we expect to have come after it? Well, we're used to seeing the 11 symbol come after the 10 symbol aren't we? However, this eleven is no longer ten plus one, but nine plus one! In our case, we will use the phrase nine and one. That makes 12 represent nine and two. It makes 13 represent nine and three (aka a dozen) and 14 represent nine and four (aka a baker's dozen).

But let's say we chose the alternative nine representation, 9 instead of 10. What would nine and one, the number after nine, look like then? Why is using 10 to represent nine and one problematic for a nonary system? Experiment until you find at least one answer and feel free to start write the resulting number line down… Well, 11 is the natural next candidate, thanks to having a one in the ones place. Naturally, nine and two would be 12, and nine and three would be 13. Now let's go ahead and compare our respective number lines for the nonary system again. We have one as: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13 14 And a second as: 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13 14

Whoa! So far we see that the second method is the same representation of the nonary system, except that nine is represented as 9 instead of a 10. Let's explore a little further.

Let's explore a little further. What comes after nine and eight(18)? Well, we'd have nen nen wouldn't we? In other words, we'd have two nines! Let's see, if nine and eight looks like 18, then surely nine and nine looks like 19? But if we use the phrase 'two nines', what would we expect the symbol look like? Well, what would we expect two 10's to look like? 20 perhaps? So, now we have two ways of describing the the number following nine and eight. We have nine and nine (19), and two nines (20). We're used to knowing the symbol 20 as twenty aren't we? Its name in the nonary system, two nines, actually looks a lot like twenty doesn't it? We notice that these representation seem to jive with our number systems. Let's take a look at our newly revised number lines for the nonary system. 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13 14, 15, 16, 17, 18, 20 And a second as: 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13 14, 15, 16, 17, 18, 19

Now, what comes after two nines (20)? Well, two nines plus one (20+1). Let's call this number two nines and one, and represent it as 21. Hm, if we had described our number as nine and nine plus one (19+1), the label for the next number isn't as obvious is it? For both symbols representations, we will use two nines and one (21) to represent the number after two nines or nine and nine. After two nines one comes two nines and two (22), then two nines and three (23), and so on. We see our new number lines as: 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25 ,26, 27, 28 And a second as: 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25 ,26, 27, 28

Now, what would be the symbol(s) that come after two nines and eight (28) if we decide to make our counting symbols consistent? That's right, three nines (30) or two nines and nine (29). Now we're really starting to see a pattern in our counting symbols. 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25 ,26, 27, 28, 30, 31, 32 And a second as: 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25 ,26, 27, 28, 29, 31, 32 Now, let's take a moment to reflect. Our two symbolic representation of our nonary system seem to be virtually identical. The only apparent difference is that multiples of nines are represented differently. For example five times nine is represented as five nines (50) or four nines and nine (49). There is another difference that you might not have considered. In our second set of representations, there is a symbol create specifically for nine, 9. In our first symbolic system, nine does not have its own unique representation. Though it is apparent that we still need nine symbols to represent this sort of base 9 system, and so zero comes in to help. In the second type of symbolic representation, there we are not utilizing the symbol for zero in for representing positive numbers. From here on out we will exclusively work with the first type of symbolic representation, where nine is represented as 10 because this representation allows us to use zero, and is closer to what we are accustomed to. We are used to using the decimal system, where the base number is represented as 10, and now we can see that the two symbolic representations are almost identical in a way.

Now, let's consider how we are representing numbers, generally speaking. Well, let's look at three nines and two (32), to start. That's really another way of saying that we have three sets of nine, and then an additional two. Say we had two dozen eggs? How many eggs is that in our nonary system? Well, we know that nine and three equals a dozen (13). To get two dozen, we can double the ones and nines place to receive two nines and six (26). To get four dozen

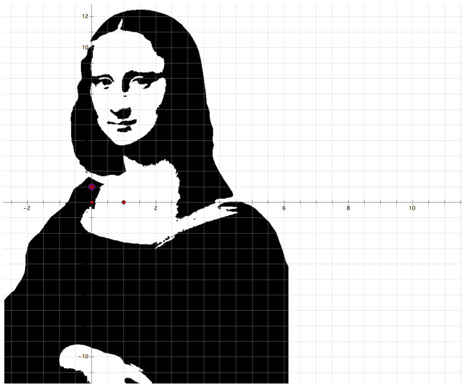

Figure 1: Mona Lisa graphical representation.

eggs, we can quadruple the ones place and nines place of thirteen to receive 4 nines and... how many ones? In our decimal system, three time four is 12. However, since multi-digit numbers mean something different in the nonary system, it requires a different multiplication table. Let's try multiplying four and three by counting by threes. Three plus three is six. Six plus three is nine. Nine plus three is nine and three. So, we see that four times nine and three (4*13) is equal to four nines plus nine and three (40+13) or five nines and three (53). This exercise may help you harken back to the days when multiplication and addition were not so trivial. This has been an exploration of the subtleties of working with different base systems. This has just been an exploration of one very close cousin to the decimal system, but there are others which are much more prolific, such as the dozenal system. In the following essay I will explore the binary system, also known as the base 2 system.

However, as you might have suspected, for uses of a relation, we wish to keep things a bit more restrictive. For example, imagine that our domain was time, our codomain was distance, and our graph was the set of ordered pairs that consisted of the time and your distance Mars is from Saturn. It wouldn't make sense for the planets to be two different distances from your home at a given time! In other words, each object in your domain(time) won't have multiple ordered pairs or distances from the house. When this happens, we say that a relation is functional or 'right unique.' In other words, for ever object in the domain, it is paired with a unique object in the codomain. Notice now that using this terminology, the domain is considered 'left' and the codomain is considered 'right'. We would also expect that Mars and Saturn were always a certain distance from eachother (even if that distance is zero). In other words, we would think that for every object in our domain(time), there is an ordered pair or distance. When this happens, we call the function 'left complete'. In other words, every object in the domain is paired with some object from the codomain. Now, when an object is both functional (right unique) and 'left complete', we call it a function! Functions are some of the most popular tools used in mathematics, and it behooves the mathematics student to be well acquainted with them.

Let's consider some other important properties that a relation might have. The objects in the codomain which are in ordered pairs can be referred to as the range. For example, let's use the last paragraph's relation, where the codomain was distance and your graph the distance of Mars from Saturn. If the closest they every got was 7 astronomical units and the farthest they ever got was just up to, but not quite 9 astronomical units, then our range would be written as [7, 9). This notation refers to every number between 7 and 9, including any number next to a bracket. So 7 would be included in the range, but 9 would not. If a relation has every object from the codomain in an ordered pair, ie. if the range is also the codomain, then it is called surjective, or right complete.

Our final property in this list will be 'left uniqueness' or injectivity. Imagine you enter a chair store. Your domain may be a height. Your codomain may be chairs in the store. Your graph ordered pairs of a height and a chair with that height. Notice that for every object in the codomain(a chair) there will be a unique object in the domain (a height). Therefore, for each object in the codomain, it has a unique object pair in the 'left' domain. Notice that if every object in the domain is paired with a unique object in the codomain (right unique or functional) and every object in the codomain is paired with a unique object in the domain (left complete or injective) then every object in either the domain or codomain is in one unique ordered pairing. We call a relation with this poperty one-to-one. The function we observed earlier, 2x=y is one-to-one. Notice how for any choice of x, there can be only one y and for any choice of y, there can be only one x.

We have gone over four properties: Left unique or injective Right unique or functional Left complete Right complete or surjective

To reiterate, if the relation has the first two properties, it is one-to-one. If it is right unique and left complete, it is a function. If it has all four properties, it is said to be a bijection. Notice that any one-to-one function can be made into a bijection, simply by restricting the domain and codomain to the what can be found in your graph's ordered pairs!