1785

The University of Georgia

Mathematics Education
EMAT 4680/6680 Mathematics with Technology
Jim Wilson, Instructor

Instructional Unit

by

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0.1 Relations: Day 1 and 2

Group Activity

Start by seperating the students into groups and ask them to think of pairs of numbers. They can be related in some way, but don't have to be. Examples of answers might be (2,5), 8 and 3. Let them know that they can be creative, and should see if they can think of ways of describing pairs without explicitly writing out the individual numbers. Creative solutions might be to say, "The lattitude and longitude of all the McDonald's stores, the solutions to x=5y, or the room number followed by the chairs." After the groups have all come up with their sets of numbers, congratulate them, and explore the different answers they have come up with, particularly any creative ones. The students answers should provide fuel for teaching moments. Let them know that that future pairs will be of the form (x, y).

Class Discussion

Next, start a whole classroom discussion on how we might represent certain pairs. If students are shy, you can start by asking about the creative solutions that individual groups came up with. You might ask them if there was a way to use a pair of number lines to represent multiple numbers, with this intention of introducing the concept of a graph. End the conversation with an explanation of how a visual graph can be used to represent different points and that the axis are x-axis and the y-axis.

GSP Activity

Have the students bring up GSP, and help them to use it to plot out the different pairs of numbers that they came up with in the group activity in a graph. They don't have to get every point plotted or be very exact with they place their points.

As a second part of the activity, ask them make an interesting picture, and come up with a way of describing the picture in terms of the points that make it up. This can be a very challenging task depending on the picture! You may have the student just pick one part of their picture to describe. One example they might choose is to draw a profile of a person, such as our Mona Lisa graph. There are a variety of ways that they might try to describe what are an infinite number of points. For example, they may describe two points that many others are between. They may even suggest equations such as y=5x to describe a line on their graph.

Group activity

Ask the students to think about what sorts of things can be represented by numbers. Also be sure to ask what is special about each of these. Can there be any number of these things? For example, can there be a negative number, a fraction, a very large number? Does it depend on what we mean by these? These are ideas that relate to completeness.

Clasrrom discussion

Give the groups some time to think about how we might use pairs to describe something.

Start the class giving the example of using pairs to represent time and location. One such example is: "John is 5 yards away 7 minutes after midnight," could be described by (7, 5). Now start a discussion over possible traits of this type of representation. You or the students may bring up several other examples, such as lattitude and longitude to represent a location. The intended take

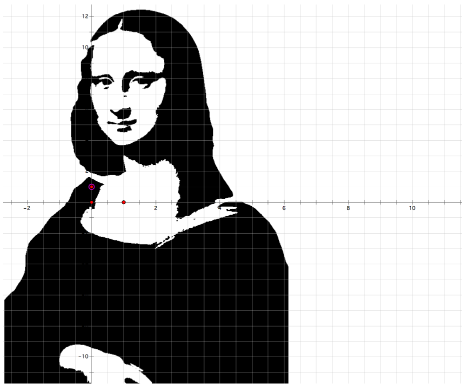

Figure 1: Mona Lisa graphical representation.

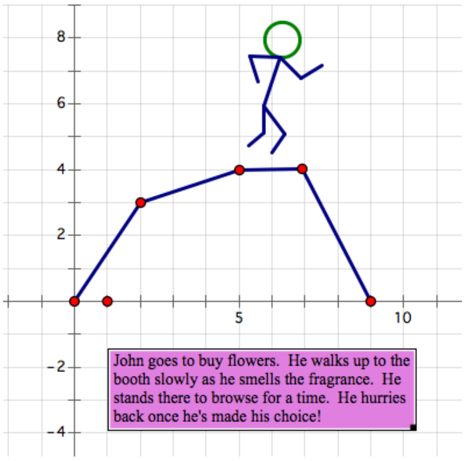

Figure 2: Example of a student's graphical representation of a story

away from this discussion is that we can relate two categories using a 'relation' that links them together. Also, depending on what your domain and codomain are some numbers may only be able to appear once, or some numbers might not be able to appear. This is intuitively getting at the idea of concepts such as injectivity and onto relations.

GSP Activity

Have the students bring up GSP, and have them come up with a story that involves someone traveling over time, using a graph to describe distance. Encourage creativity. This will be turned in as their assignment for the day. If there's time, ask for a volunteer to give their story, using their graph as an assist. This will be used a discussion point. See figure 2.

0.2 Functions: Day 3 and 4

Class Discussion

Begin with a review of the previous class, discussing what we learned about

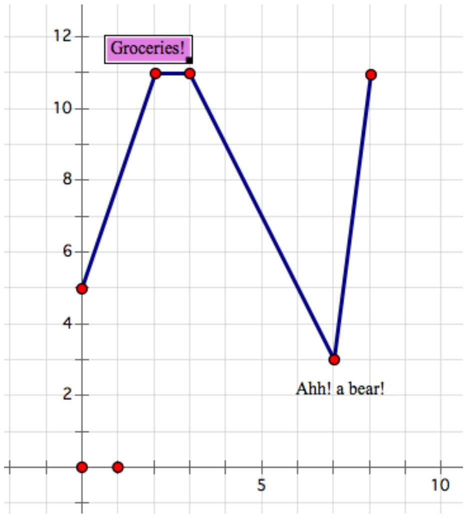

Figure 3: Example of a distance from home graph following the farmer's story.

relations and how some relations don't allow every pair of data, or only allow some numbers to appear once in the domain or codomain. Have at least one student give their story that is supplemented by their graph. Start a discussion about why this graph is a certain way, based on the story. The big provocative question might be, "why is there no point above another point?". Or "Why is the graph a continuous, unbroken line?" "Could we think of an example when either of these things wasn't the case?" Teleportation, being in two places at once, and 'temporarily not existing' are examples of responses." The take away from this discussion is that certain types of relations have the properties of functions and that we will mostly consider this type of equation from here on out.

GSP Activity

Ask the students work on making a graph that relates to a story: The farmer starts 5 miles from home. He takes about 2 hours to walk another 6 miles away from home to get milk from the store. He then shops there for about 1 hour. Then after he's been walking for four hour toward home, and is only 3 miles from home, he sees a bear! He runs away, getting 8 miles farther from home in one hour! Note that this can be a prime example of a teaching moment. Also recall that the graphs will look different depending on whether the student is graphing the distance traveled or the location from home! See figure 3 for a distance from home graph of this situation..

Class Discussion

Have students volunteer to describe their representation of the story to the class. What were the two axis of the graph? Did the student choose the distance variable to be distance from home or total distance traveled? What does a horizontal line mean? Is there a way to determine the farmers average speed at a given time? Where is the y-intercept and what does it mean? Is the graph entirely composed of straight lines? Any errors that are made are ample teaching moments or seeds for class discussion.

Now, if it does not come up in discussion, make sure to use this as an oppurtunity to discuss how to find a the farmer's speed. This is when we discuss the idea of slope and average rate of change.

GSP Activity

Give the students a variety of equations. y=5x, y=x, y=0x, y=0, y=1/3x, y=2.5x

Ask the students to do their best to graph these using the gsp line tools.

Class Discussion

Now ask the class to imagine a set of pairs, where one number is always twice as large as another number. What would that look like on a graph? After some thoughts and allowing some discussion and writing, reveal that this can be represented as the function y=2x or y=1/2x. Discuss the properties of this sort of function. What does it look like when represented by a graph? If they say it is a line, ask why it is a line. How could we take an equation like y=5x and determine what the graph is without thinking up many pairs? What does this type of graph have in common with our farmer's graph, and how is it different?

GSP Activity

Give the students a variety of equations. y=5x, y=x, y=0x, y=0, y=1/3x, y=2.5x

Ask the students to do their best to graph these using the gsp line tools. They may start with only graphing individual pairs, but eventually to try and graph the general line!

After the students have spent some time working on coming up with pairs and lines, we ask them a thought provoking question: what would a function would look like if 'y' was always the same value but x could be any value? Can you graph this kind of relation? Is there more than one of this kind of relation and what do they seem to have in common?

Class Discussion End with a student let discussion on relations, properties of their graphical representations, and why they might be useful for understanding real life issues. If there is time, have the students think up situations that might be modeled using the equations discussed in class or problems that could be addressed. Let the students know that they don't have to know how to solve these problems, just that relations might be helpful in looking for a solution. In the next class, students will learn about the general forms of linear functions.

0.3 Linear Functions and function addition: Day 5 and 6

Class Discussion

Have the students form groups of three and have them talk to eachother about what their are on relations, functions, and graphs. These don't have to be explicit definitions or mathematical properties, but each student should participate giving their own interpretations. Encourage the students when they make their discussions.

After some group discussion, return to a full class discussion and ask the students what are some real life situations that can be modeled. As ideas are being described, ask some questions to get the students thinking. For example, one example that of a situation that can be modeled by a function is 'water in a glass over time.' Some questions that could be asked of application of function are: What shape might the graph of this function have? Given certain shapes of the function, what could be happening to the graph? Is there a way to determine how fast the water is coming out or into the glass? What would the function look like if the amount of water in it stayed the same? What would be the explicit function for this situation, if the y-value was the number of ounces in the glass and the x-value was a given time?

Group Activity

Have the students get into groups and give them a large glass and a washable marker. Have them come up with a story for the glass that takes place over the course of six minutes. Have them mark seven spots on the cup, each corresponding to how full the cup is in their story at 0 minutes, 1 minute, 2 minutes, and so on. As you walk around the room, ask the students to share their story, ensuring that they understand the spirit of the activity.

GSP Activity

Now, have the students plot the seven points they have chosen on GSP. Now, remind the students that there is water in the cup all the time in their story, not just at the minute marks. Have the students discuss how they might determine how much water was in the glass inbetween the minute marks.

Have the students discuss how they might describe the points inbetween the zero minute mark and the one minute mark. You might get a response that they are the points on the line connecting them.

Group Discussion

Now, ask the students, "what if this line connecting the zero minute and the one minute mark went on forever?" Does that seem like any sort of function we have discussed before? Remind the students of the previous class activity that involved determining linear functions, such as y=5x. Give the students some time to think about what sort of linear function the line between the zero minute mark and the one minute mark is. What makes this question challenging, is that this is the first time in class that they are working with a linear function that has both a slope and y-intercept that are non-zero.

There are a variety of responses you might get, such equations that consider the slope but not the y-intercept. As a hint, you might ask what the equation looks similar to, and how this graph, with the y-intercept, is different. Once this question has been resolved, discuss with the class what a y-intercept is and what it means in the context of the glass example.

Now ask the students how much the water changed in their linear function. Is this same know matter which minute we look at? (We are still looking at the linear function based on the first two points) What relationship does this have with the equation of the line? After some discussion, let the class know that this value is also known as the slope, and is the coeficient on the x term in linear function.

GSP Activity

Have the students think of new zero and one minute marks that they can use to define a linear function. Have them graph their line. Then ask them what other pairs of points are on the line. Eventually, ask them if they can think of a way to come up the y-intercept form of the function just from knowing the zero minute and one minute values. Can they give describe an assortment of pairs from the function when they only know the y-intercept form of the function. Ask the students to discuss with a partner on why they are able to do this.

Class Discussion

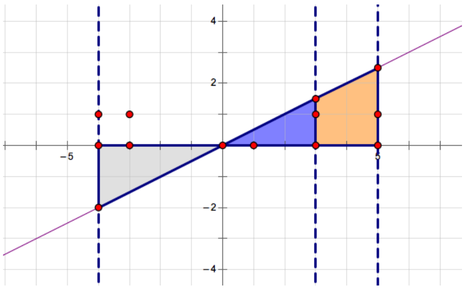

Figure 4: Shapes found by putting vertical lines through a given function.

Start a class discussion. Ask the students how they might describe the amount of water in two glasses. Do they know how they would create a function that describes how much is in two glasses from the zero minute mark to the one minute mark? What if the second glass was empty? What if the second glass always had the same amount of water in it? What if the second glass started empty, and had water poured into it at the same rate as the first glass? What if water was poured in at twice the rate?

GSP Activity

Have the students come up with pairs of functions for glasses with water in them and have the functions graphed simultaneously on GSP. Color coding can help with making the graphs distinguishable. Then have the students graph what they determine to be the function of the amount of water in both cups combined. They may experiment with how they do this, including simply adding up the amount of water in the two glasses at a give time to get a point in the graph. After most of the students have made headway on doing this for at least on pair of functions, encourage them to see if there is a general way to do this, when you just want to use the y-intercept forms of the functions without graphing them. How might they determine the function of the combined water in the cups in a simple way? A wrap up discussion of this process concludes the students' introduction to function addition.

0.4 Geometric approach to quadratic equations: Day 7 and 8

GSP Activity Have the class think about the types of functions that they have looked at so far. If possible, have them open up their old saved GSP work. Now ask them what would happen if you placed a vertical line somewhere on the graph. Would there be any shapes created? Ask that they put down one, two, or three vertical lines on one of their graphs, and ask they they look to see how many shapes they can discover. The shapes are those bounded by the function, the vertical lines the students place, and the x-axis. They are likely to find some combination of triangles, rectangles, and trapezoids. Have them color in the shapes that they find. See figure 4.

Now, have the students recall how to find areas of shapes. Are they able to find the areas of some of the shapes that they made on their graph? Assist the students who are having a hard time getting started.

Class Discussion

Have the class discuss their methods for solving the polygons they made in their graphs. Three types of shapes should have come up, triangles, trapezoids and squares. The discussion should describe what resulted in each of these shapes, and students should be welcomed to volunteer the strategiest they used to find the areas. Discuss as a group how you can easily find the area of a rectangle in these graphs, since it is simply the distance between two vertical lines and the distance from the function to the x-axis.

Group Discussion

Now have the students discuss amongst themselves in a group what the procedure is for solving the area of a polygon bounded the y=2x the x-axis, and an arbitrary vertical line, x=a. Ask them how this might change if the slope was a different number, like six? Can they find a similar procedure for a polygon bounded by the function y=2x+3, the x-axis, y-axis and x=a? If students are having trouble, remind them that this sort of trapezoid is really just a triangle on top of a rectangle. As a bonus question for students that get this far, ask if they can describe the general procedure if the equation was an arbitrary y-intercept and slope greater than zero. Alternatively, the students could plot points on the graph that represent the area of the polygon for certain values of a.

GSP Activity

Discuss the generalized methods that they have found. Lead the discussion toward resolving the general form of this equation, ie., that the shape of the polygon bounded by the x-axis, y-axis, y=mx+b, and x=a is $\frac{m}{2}a^2 + ba$. Notice that polygons underneat the y-axis are considered to have negative area!

Now using this formula have the students plot various a values with their corresponding polygon areas on the same graph as the linear functions. They have the freedom to choose their linear function. Once they have plotted a sufficient number of points, ask them to describe the shape of their new graph? It appears to be that of a parabola! At this point, introduce the plot new function option in GSP. Have them put $\frac{m}{2}a^2 + ba$ into the function, and show how it lines up with their points. At this point, it is revealed to them, that the shapes of the areas underneath the graphs of linear functions actually represent the set of polynomials known as quadratics! Have the students experiment some with us, finding the equations for the area underneath a variety of functions.

Class discussion

We have shown the class how quadratics have emerged out of linear equations. This demonstrates one interpretation of quadratics. Start a discussion of the various uses or interpretations of quadratics. What does the class think of the shape. What would a story be like if this was a story's distance function? How would the person's speed be changing?

0.5 Slope and area interpretations: Day 9 and 10

Class Discussion

Remind the class that we have recently described how to find the area underneath an linear function by looking at it as shapes. Start a board problem with class participation. This problem is to find the area of the shape between the x-axis, the y-axis, $y = \frac{1}{2}x + 6$, and x=6. Encourage students to solve it in multiple ways. End by solving this using the general formula, $\frac{m}{2}a^2 + ba$. Do another problem such problem except where the linear function is y=2x-6. Do the same process, but once you get to using the generic formula, $\frac{m}{2}a^2 + ba$, explain as you go through the problem why the triangle beneath the x-axis is treated as having negative area, similar to how the y-values are negative beneath the x-axis. Explain that from now on, these polygons beneath the x-axis are considered to have negative area.

Group Discussion

Now we provide groups a worksheet with a series of questions. Once a group feels satisfied with one question, they move onto the next. The group should also attempt to represent each question as a function, where the y-axis is the speed of the farmer. If a farmer travels at 4miles per hour for an hour, how far does he go? If he travels 4 miles per hour for 8 hours? What if traveled at miles per hour 2 miles per hour the first 4 hours, then 6 miles per hour the next four hours? What if he travels at 0 miles per hour the first hour, and increases his pace by one mile per hour each hour, so that his second hour he goes one mph and his third he goes two mph and so on? What if his speed increases at a constant rate from zero to 8 miles per hour over the eight hours. Intuitive agruments are okay. Now take each speed function that was made for these graphs, and see if the students are able to make distance functions. How are they the same? How are they different? This may take some time, and the students may need some assistance incorporating the previous ideas of finding the area underneat a function. This estabilished the idea that taking the area underneath a velocity graph creates the corresponding distance graph.

Class Discussion

Come together as a class and discuss the solutions that the groups had to the questions, and the graphs that they developed. You might ask the students if there seems to be a pattern emerging in the graphs as they change. Note that there seems to be a pattern of the speed graphs having more and more steps, until the last graph is just a slope. You might ask what it would look like if we just kept increasing the number of 'steps' that the graph had. The conclusion being, that it would look more and more like the last graph. Explain the interpretation of the pairs of class, and talk about why it makes sense that the area underneath the speed graph is the distance graph. Note that the initial two questions on the Group discussion worksheet are meant to provide the initial intuitive understanding of this, "If a farmer travels 4 miles per hour for an hour, how far does he go?"

GSP activity

Now that the class is overall on the same track, have them use GSP to graph their distance functions. Now have them find the slope of the graphs at various points. Have them plot the slope on the same graph, with the point representing (a, slope at a), where a is an arbitratry point. Ask them if they can find rules of thumb that plot the slope of every point on each graph. Now once they have made decent headway on this, ask them what the slope represents in this case. If they recall from a previous class, since slope is the change in y-values over the change in x-values, it essentially take on the unit of y-values divided by the unit of the x-values. In this case, that is miles over hours, or miles per hour. So, what they have really been doing is plotting points on the speed graph! The big reveal will be comparing the points plotted for the slopes with their original speed graph, and seeing them match up. Note, there is not expectation of students successfully plotting points for the slope of the last graph, just that they think about it.

Class Discussion

Come back to a class discussion of the students' findings and what they thought of the surprising connections between the speed graphs and the distance graphs. Discuss the reasons why the area underneath a function determines a speed graph's distance graph and why the slope of a function determines a distance graphs speed graph. Also discuss the last graph, that involved a speed graph with a slope of one. It turned out that the distance graph was a parabola. The class wasn't able to find the graph's slope because it didn't have a constant one. Talk about why it's speed graph makes sense, given the other sets of graphs. You may also discuss, how the students might attempt to find the speed graph if it was not already available to them. Suggestions might include using tangent lines. Be sure to open up the discussion for any questions on the subject for students to share their points of view on the subject.

Optional

Further explore the last graph with the distance graph in the form of a quadatic. Explore with the students why its speed graph determines it's instantaneous rate of change. Ask the student how they might determine this instaneous rate of change for other quadratic equations. They might recall the general form of the area underneat a curve, and understand the connection, in which case they merely need to reverse engineer this generalized rule. If not, the teacher may ask leading questions to talk about this option. You may also explore the slopes of speed graphs and why these lead to acceleration graphs.

From here there are many extensions available to you. You may continue to work with quadratics. You may ask create a distance problem to determine when a person is fartherst or closest to you. This allows you to bring up ideas such as minimums, maximums, and that these must occur when the corresponding speed function is zero. The class may work with looking at a linear functions in their point slope form. The class may also explore solving the zeros of quadratic equations.

Conclusion

This has been an introduction to functions through the lense of relations. This function introduction is designed to act as a precursor to some of the more accessible ideas within calculus to create a foundation for future success through mathematical discourse and argumentation.