The University of Georgia

Mathematics Education
EMAT 4680/6680 Mathematics with Technology
Jim Wilson, Instructor

The Simson Line

by

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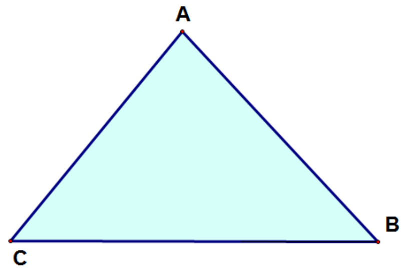

Figure 1: Triangle ABC

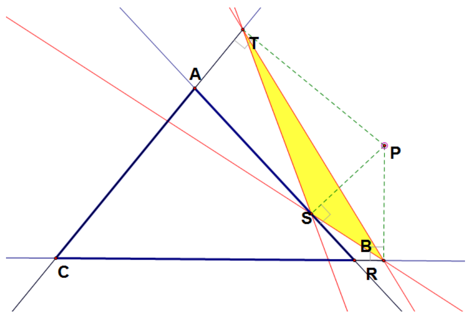

Figure 2: A pedal triangle for triangle ABC.

Simson Line

Consider the Triangle ABC from figure 1. We would like to find the point(s) 'P' that generate a degenerate pedal triangle for triangle ABC. A degenerate triangle is one which has collinear vertices. The line which intersect these three vertices is known as a simson line.

An example of a pedal triangle for triangle ABC is shown in figure 2. We see that this triangle is not degenerate. We also observe that in order for this to be a degenerate triangle, R and T will have to be on opposite sides of line AB if line AB is closes to our pedal point. This gives us some sense of how 'close' our pedal point has to be to ABC for the pedal triangle to be degenrate. With some experimentation, it becomes apparent that all degenerate pedal triangles are the same distance from the circumcenter of the triangle ABC. In fact, the pedal triangle will be degenerate if the pedal point is on the circumcircle of the triangle. See figure 3.

We will use figure 3 for our proof. We may assume that unless the pedal point is on a vertice, the properties found will remain the same if the points are relabeled. For example, T and R will be on opposite sides of AB. This allows us to use figure 3 for a general

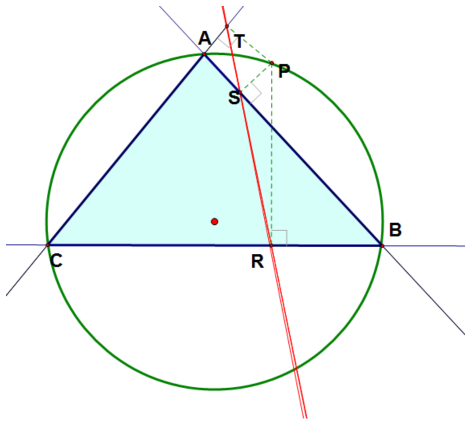

Figure 3: A degenerate pedal triangle whose pedal point is on the circumcircle

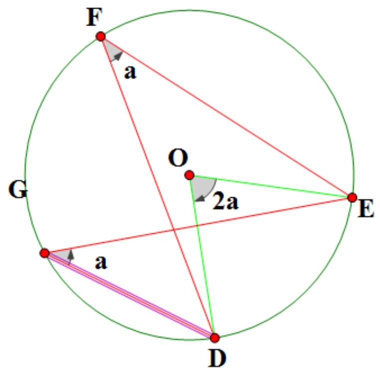

Figure 4: Inscribed angle theorem

proof. More specifically, we will show that ∠TSR is equal to 180 degrees. If so, then our degenerate triangle or simson line exists. The general strategy for this proof will be to begin by showing that ∠PST is supplementary with ∠CAB. We will then show that ∠CAB is equivelent to ∠PSR, proving our conjecture. Prior to this, we will have recall the necessary background material on inscribed angles and cyclic quadrilaterals. Recall the inscribed angle theorem illustrated in figure 4. If the points D, E and F are on a circle, then ∠DOE= 2∠DFE. Note that this implies that for any other point G on the circle, ∠DFE = ∠DGE.

Now consider figure 5. We see that if H is on the circle and DHEF forms a quadrilateral, then applying the inscribed angle theorem shows that ∠DHE is 180-a. Therefore, if a quadrilateral is cycle, it opposite angles must sum to 180 degrees.

Now observe in figure 6, that ∠PTA and ∠PSA are both right

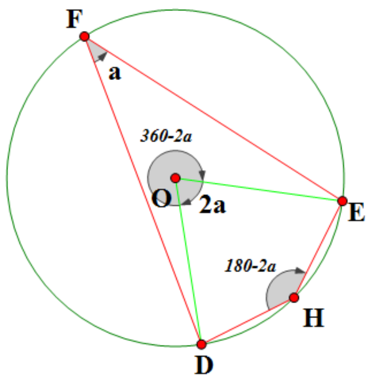

Figure 5: Inscribed angle theorem applied to cyclic quadrilaterals

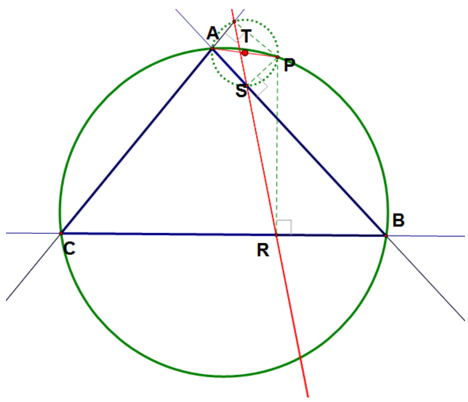

Figure 6: Inscribed angle theorem applied to cyclic quadrilaterals

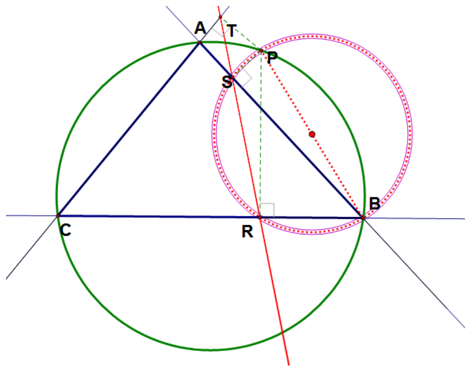

Figure 7: BSRP is a circular quadrilateral

angles and therefore sum to 180 degrees. Therefore, the quadrilateral must have a circumcircle. By the inscribed angle theorem, this also means that $\angle$TAP and $\angle$TSP must be the angle measure. Since $\angle$CAP + $\angle$TAP are supplemental, so must $\angle$CAP and $\angle$TSP.

We shall now show that $\angle$CAP has the same angle measure as $\angle$PSR. We first observe that $\angle$PSB and $\angle$PRS are both right angles. Therefore, both R and S are located on a circle whose diameter is PB. See figure 7. So, PBRS is a cyclic quadrilateral. Therefore, $\angle$PSR and $\angle$PBR are supplementary. Since APBC is also a cyclic quadrilateral, $\angle$PBR (or $\angle$PBC) and $\angle$CAB are also supplemental. Since $\angle$PBR is supplemental with both $\angle$CAP and $\angle$PSR, these two angles must have the same measure.

Recall that $\angle$PST is supplemental with $\angle$CAP. Since $\angle$CAP has the same degree measure as $\angle$PSR, $\angle$PST and $\angle$PSR are supplemental. Therefore, $\angle$TSR must be 180 degrees, and T, S and R are collinear. Therefore, we see that the simson line exists if the pedal point is on the circumcircle of the triangle.