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The Fibonacci sequence

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Generate a Fibonacci sequence in the first column using f(0) = 1, f(1) = 1,

$$f(n) = f(n-1) + f(n-2)$$

a. Construct the ratio of each pair of adjacent terms in the Fibonacci sequence. What happens as n increases? What about the ratio of every second term? etc.

b. Explore sequences where f(0) and f(1) are some arbitrary integers other than 1. If f(0)=1 and f(1) = 3, then your sequence is a Lucas Sequence. All such sequences, however, have the same limit of the ratio of successive terms.

The Fibonacci Sequence is a series of numbers defined by f(n) = f(n-1) + f(n-2), where the next number n is defined by the sum of the previous two numbers n-1 and n-2.

Below we can see a table of values for the series defined by:
f(n) = f(n-1) + f(n-2), where f(0)=1 and f(1)=1.

The table also displays the ratio of each pair of adjacent terms in the sequence defined by $\frac{f(n)}{f(n-1)}$; the ratio of each pair of every second term defined by $\frac{f(n)}{f(n-2)}$; and the ratio of each pair of every third term defined by $\frac{f(n)}{f(n-3)}$.

n	f(n)	$\frac{f(n)}{f(n-1)}$	$\frac{f(n)}{f(n-2)}$	$\frac{f(n)}{f(n-3)}$
0	0			
1	1			
2	1	1		
3	2	2	2	
4	3	1.5	3	3
5	5	1.666666667	2.5	5

6	8	1.6	2.66666667	4
7	13	1.625	2.6	4.3333333
8	21	1.615384615	2.625	4.2
9	34	1.619047619	2.61538462	4.25
10	55	1.617647059	2.61904762	4.2307692
11	89	1.618181818	2.61764706	4.2380952
12	144	1.617977528	2.61818182	4.2352941
13	233	1.618055556	2.61797753	4.2363636
14	377	1.618025751	2.61805556	4.2359551
15	610	1.618037135	2.61802575	4.2361111
16	987	1.618032787	2.61803714	4.2360515
17	1597	1.618034448	2.61803279	4.2360743
18	2584	1.618033813	2.61803445	4.2360656
19	4181	1.618034056	2.61803381	4.2360689
20	6765	1.618033963	2.61803406	4.2360676
21	10946	1.618033999	2.61803396	4.2360681
22	17711	1.618033985	2.618034	4.2360679
23	28657	1.61803399	2.61803399	4.236068
24	46368	1.618033988	2.61803399	4.236068
25	75025	1.618033989	2.61803399	4.236068
26	121393	1.618033989	2.61803399	4.236068
27	196418	1.618033989	2.61803399	4.236068
28	317811	1.618033989	2.61803399	4.236068
29	514229	1.618033989	2.61803399	4.236068
30	832040	1.618033989	2.61803399	4.236068

As n increases, the ratio of each pair of adjacent terms in the sequence defined by $\frac{f(n)}{f(n-1)}$ converges to the Golden Ratio, or Phi (φ). We can represent this ratio as the following limit:

$$\mathbf{Lim}_{n\to\infty} \frac{\mathbf{f(n)}}{\mathbf{f(n-1)}} = \boldsymbol{\varphi}$$

Which is the same as saying the following:

$$\text{Lim}_{n\to\infty} \frac{f(n-1)+f(n-2)}{f(n-1)} = \varphi$$

$$\text{So, as } n\to\infty, \frac{f(n-1)+f(n-2)}{f(n-1)} = \varphi,$$

$$\frac{f(n-1)}{f(n-1)} + \frac{f(n-2)}{f(n-1)} = \varphi,$$

$$1 + \frac{f(n-2)}{f(n-1)} = \varphi,$$

$$\frac{f(n-2)}{f(n-1)} = \varphi - 1,$$

$$\frac{f(n-1)}{f(n-2)} = \frac{1}{\varphi-1}.$$

As $n \to \infty$, $\frac{f(n-1)}{f(n-2)} = \frac{f(n)}{f(n-1)}$.

Which suggests:

$$\frac{f(n)}{f(n-1)} = \frac{1}{\varphi - 1},$$

$$\boldsymbol{\varphi = \frac{1}{\varphi - 1}}.$$

We can now solve for φ:

$$\varphi * (\varphi - 1) = 1,$$

$$\varphi^2 - \varphi - 1 = 0,$$

$$\boldsymbol{\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618}$$

Which is the limit that we see in the table. The negative solution $\frac{1-\sqrt{5}}{2}$, is not a possible solution because it is negative and all of the terms in the sequence must be positive.

What about the ratio of each pair of every second term defined by $\frac{f(n)}{f(n-2)}$? From the table, it appears as though the ratio converges to φ+1.

Well, as $n \to \infty$, $\frac{f(n)}{f(n-2)} = \frac{f(n-1)+f(n-2)}{f(n-2)} = \frac{f(n-1)}{f(n-2)} + 1 = \frac{f(n)}{f(n-1)} + 1 = \varphi + 1$, as we suspected.

What about the ratio of each pair of every k term defined by $\frac{f(n)}{f(n-k)}$?

Well, by our recursive formula f(n) = f(n-1) + f(n-2),

as $n \to \infty$, $\frac{f(n)}{f(n-k)} = \frac{f(n-1)}{f(n-k)} + \frac{f(n-2)}{f(n-k)} = \frac{f(n)}{f(n-(k-1))} + \frac{f(n)}{f(n-(k-2))}$.

$$\text{So, } \lim_{n\to\infty} \frac{f(n)}{f(n-k)} = \lim_{n\to\infty} \frac{f(n)}{f(n-(k-1))} + \lim_{n\to\infty} \frac{f(n)}{f(n-(k-2))}.$$

So, let's check if this is true for the ratio of each pair of every third term defined by $\frac{f(n)}{f(n-3)}$.

As $n \to \infty$, $\frac{f(n)}{f(n-3)} = \frac{f(n)}{f(n-2)} + \frac{f(n)}{f(n-1)} = \varphi + 1 + \varphi = 2\varphi + 1 \approx 4.236$, as shown in our table.

Now let's explore a Lucas Sequence, or a Fibonacci series defined as f(n) = f(n-1) + f(n-2), where f(0)=1 and f(1)=4.

n	f(n)	$\frac{f(n)}{f(n-1)}$	$\frac{f(n)}{f(n-2)}$	$\frac{f(n)}{f(n-3)}$
0	1			
1	4	4		
2	5	1.25	5	
3	9	1.8	2.25	9
4	14	1.555556	2.8	3.5

5	23	1.642857	2.555556	4.6
6	37	1.608696	2.642857	4.111111
7	60	1.621622	2.608696	4.285714
8	97	1.616667	2.621622	4.217391
9	157	1.618557	2.616667	4.243243
10	254	1.617834	2.618557	4.233333
11	411	1.61811	2.617834	4.237113
12	665	1.618005	2.61811	4.235669
13	1076	1.618045	2.618005	4.23622
14	1741	1.61803	2.618045	4.23601
15	2817	1.618036	2.61803	4.23609
16	4558	1.618033	2.618036	4.236059
17	7375	1.618034	2.618033	4.236071
18	11933	1.618034	2.618034	4.236067
19	19308	1.618034	2.618034	4.236068
20	31241	1.618034	2.618034	4.236068

We can see from the table that the ratios remain the same as in the original Fibonacci sequence. The ratios will always remain the same no matter what two values you chose as f(0) and f(1).

Why? Do the first two terms really even matter? No, because this is a recursive relationship referring to the limit as n approaches infinity. The sum of the previous two numbers will continue to grow to infinity, no matter what two numbers you begin with. In addition, the ratios explained before will continue to approach the same limit no matter what these numbers are. So as long as the recursive relationship continues, the limit will still remain the same because the relationship between each number in the sequence has remained the same.

In addition, we have changed the beginning terms of the sequence by a constant number, keeping the form of the original function f(n) the same (as opposed to squaring the function). Therefore, the ratios will remain the same because the properties and form of the function have not changed.