More Exploring with Parametric Curves

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Objective: Review the meaning of a parametric curve. Look at an example of the sine and cosine functions and explore how changing values will affect the resulting parametric curve.

In these problems we are using parametric equations. A parametric equation is when we have introduced a parameter, $t$, and define the curve as follows: $x=f(t)$ and $y=g(t)$. A parametric curve is a collection of points of the form $(x,y)=(f(t),g(t))$, where $t$ can vary.

Now we can explore the following parametric equations for different values of $a$ and $b$:

$$
\begin{align*}
  x &= \cos(at) \\
  y &= \sin(bt) \\
  f &or \ 0 \leq t \leq 2\pi
\end{align*}
$$

We can first note that if we look at the case where $a=0$ and $b=0$, we will simply have the point $(1,0)$ because $x=1$ and $y=0$.

Next we can look at the case where both $a$ and $b$ equal 1. (Below is the parametric curve.)

$$
\begin{bmatrix}
  x \\
  y
\end{bmatrix} =
\begin{bmatrix}
  \cos(t) \\
  \sin(t)
\end{bmatrix}
$$

From the graph we see that when $a=1$ and $b=1$ we get a circle of radius 1.
Now if we vary $a$ and $b$ and keep them equivalent, we get the graphs below.

\[
\begin{align*}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \cos(4t) \\ \sin(4t) \end{bmatrix}
\end{align*}
\]
From these curves we see that if $a$ and $b$ are equivalent, then we always have a circle of radius length 1. The $a$ and $b$ multiplied by parameter $t$ merely change the number of revolutions completed. Therefore, if $a$ and $b$ are equivalent, which we can call $n$. The parameter $n$ can vary to be any positive or negative number. If $n$ is a positive number then the circle will be traced in a counterclockwise direction. However, if $n$ is negative, then the circle will be traced in a clockwise direction.

What if $b$ is greater than $a$? Below is a graph of several parametric equations where $a < b$.

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
\cos(t) \\
\sin(2t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
\cos(2t) \\
\sin(3t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
\cos(t) \\
\sin(3t)
\end{bmatrix}
\]

Here we can see the parametric curves begin to intersect and form “loops”. There seems to be a pattern between the number of loops and the ratio of $b$ to $a$. When $a=1$ and $b=2$, we see 2 loops. Likewise, when $a=2$ and $b=3$, we see $3/2$ loops, and when $a=1$ and $b=3$ we see 3 loops.
Now, what if $a$ is greater than $b$? Below is a graph of several parametric equations where $b < a$.

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} = \begin{bmatrix}
    \cos(2t) \\
    \sin(t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} = \begin{bmatrix}
    \cos(3t) \\
    \sin(2t)
\end{bmatrix}
\]

\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} = \begin{bmatrix}
    \cos(3t) \\
    \sin(t)
\end{bmatrix}
\]

This relationship is not quite as easy to see. However, we can still draw several conclusions in this case when $b < a$. If $a/b$ is an odd integer, then the parametric curve is a vertical version of when $a < b$. If $a/b$ is an even integer the parametric curve does not intersect. The value for $a/b$ provides the number of times that the curve crosses the $y$-axis. In the examples above, when $a=2$ and $b=1$, we get an image that looks like a “sideways parabola”, it is a curve that does not intersect, crosses the $y$-axis twice and the $x$-axis once.