The University of Georgia

Mathematics Education
EMAT 4680/6680 Mathematics with Technology
Jim Wilson, Instructor

Exploring the Orthocenter of a Triangle

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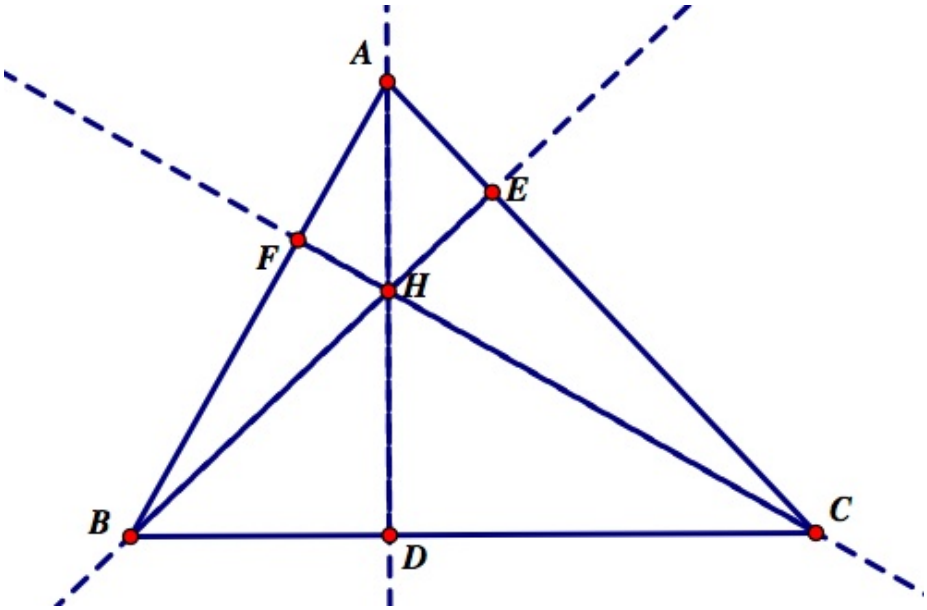
Objective: Given the acute triangle ABC, construct the orthocenter H. Let points D, E, and F be the feet of the perpendiculars from A, B, and C respectfully.

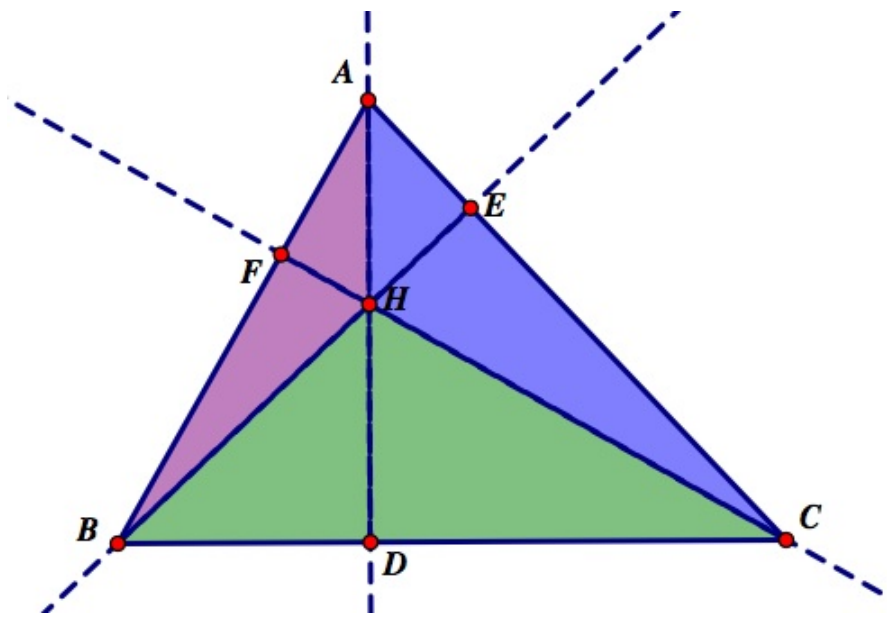
Below is an image of an acute triangle ABC, with an orthocenter H. The points D, E, and F are the feet of the perpendiculars from A, B, and C.



First, we will prove the following statement:

$$\frac{HD}{AD}+\frac{HE}{BE}+\frac{HF}{CF}=1$$

This equation suggests that the summation of the ratio of the sides of the three similar triangles is equal to one. We are going to complete this proof by looking at the area of three triangles. The total area of a triangle is given by the formula ½(base)(height). Therefore, we can represent the area of ΔABC as ½(AB)(CF) or ½(AC)(BE) or ½(BC)(AD). We can also look at this triangle as the sum of the areas of three smaller triangles that comprise triangle ABC, which can be seen in the image below.



Therefore, we can say Area(ΔABC)=(1/2)(AB)(HF)+(1/2)(AC)(HE)+(1/2)(BC)(HD). Now we can do the following:

$$\frac{Area(ABC)}{Area(ABC)}=\frac{\frac{1}{2}(AB)(HF)+\frac{1}{2}(AC)(HE)+\frac{1}{2}(BC)(HD)}{Area(ABC)}$$

$$1=\frac{\frac{1}{2}(AB)(HF)}{Area(ABC)}+\frac{\frac{1}{2}(AC)(HE)}{Area(ABC)}+\frac{\frac{1}{2}(BC)(HD)}{Area(ABC)}$$

$$1=\frac{\frac{1}{2}(AB)(HF)}{\frac{1}{2}(AB)(CF)}+\frac{\frac{1}{2}(AC)(HE)}{\frac{1}{2}(AC)(BE)}+\frac{\frac{1}{2}(BC)(HD)}{\frac{1}{2}(BC)(AD)}$$

$$1=\frac{HF}{CF}+\frac{HE}{BE}+\frac{HD}{AD}$$

Now, while continuing to look at the original acute triangle, we can prove the statement:

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2$$

We can use the following equation since we have already proved it in the above section:

$$\frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} = 1$$

Now, we can manipulate this equation to represent the fact that a line segments can be viewed as the sum of its parts.

$$1 = \frac{CF - CH}{CF} + \frac{BE - BH}{BE} + \frac{AD - AH}{AD}$$

Using algebra, we can do the following:

$$1 = \frac{CF}{CF} - \frac{CH}{CF} + \frac{BE}{BE} - \frac{BH}{BE} + \frac{AD}{AD} - \frac{AH}{AD}$$

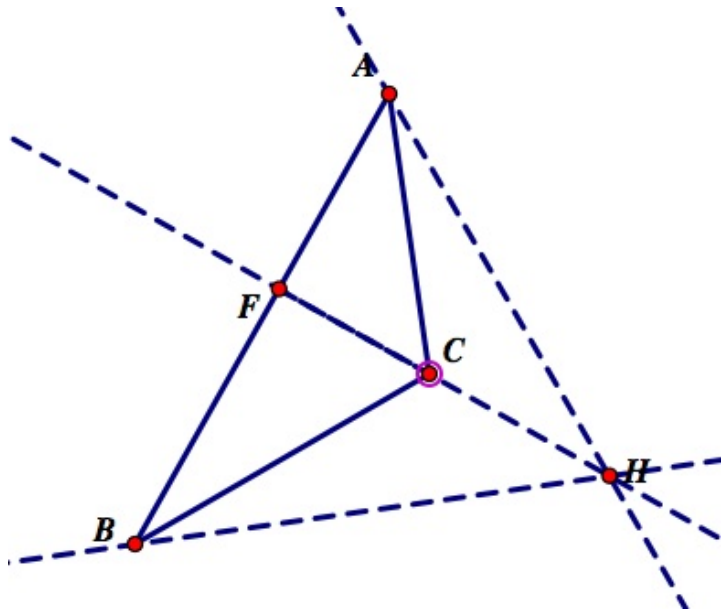
$$1 = 1 - \frac{CH}{CF} + 1 - \frac{BH}{BE} + 1 - \frac{AH}{AD}$$

$$1 = 3 - (\frac{CH}{CF} + \frac{BH}{BE} + \frac{AH}{AD})$$

$$-2 = -(\frac{CH}{CF} + \frac{BH}{BE} + \frac{AH}{AD})$$

$$2 = \frac{CH}{CF} + \frac{BH}{BE} + \frac{AH}{AD}$$

We have proved these two equations with an acute triangle and its orthocenter. Below is an image of an obtuse triangle with an orthocenter H.



When working with obtuse triangles the orthocenter can be found lying outside of the triangle. Since the orthocenter lies outside of the triangle we do not have the areas of three smaller triangles divided by the orthocenter like we did when working with the acute triangle. Therefore, we cannot use our original argument for the first proof and apply it to an obtuse triangle. Consequently, our second proof would not hold with obtuse triangles either since it relies on our first proof.