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A Collective Case Study of the Influence of Teachers' Beliefs and Knowledge on Error-Handling Practices During Class Discussion of Mathematics

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This collective case study examines the influence of 4 third-grade teachers' beliefs and knowledge on their error-handling practices during class discussion of mathematics. Across cases, 3 dimensions of teachers' error-handling practices are identified and discussed in relation to teacher beliefs and knowledge: (a) intentional focus on flawed solutions in class discussion, (b) promotion of conceptual understanding through discussion of errors, and (c) mobilization of a community of learners to address errors. Study findings suggest that, although teachers' ways of handling student errors during class discussion of mathematics are clearly linked to both teacher beliefs and teacher knowledge, some aspects of teacher response are more strongly linked to knowledge and others are influenced more by beliefs.

Key words: Discourse analysis; Qualitative methods; Reform in mathematics education; Teacher beliefs; Teacher knowledge; Teaching practice

The reform agenda in mathematics education, exemplified by the National Council of Teachers of Mathematics (NCTM) Standards documents (1989, 2000), challenges teachers to develop discourse-rich mathematics learning communities in which a significant site of student learning is group discussion of student-generated solutions to mathematics tasks. Teachers are charged with orchestrating these discussions so that students share multiple problem-solving strategies, analyze relations among strategies, and explore contradictions in students' ideas to provide greater insight into the mathematical focus (Franke, Kazemi, & Battey, 2007). Classroom norms in reform-oriented classrooms emphasize discussion, collaboration, and negotiation as ways of fostering shared meaning among a community of learners (Cobb, Boufi, McClain, & Whitenack, 1997). Research suggests that,

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within such discussions, focusing on students' errors is a particularly productive platform for stimulating student thinking about mathematics concepts and procedures (Borasi, 1994; Kazemi & Stipek, 2001). Yet, there is some evidence that U.S. teachers tend to limit inclusion of students' errors in group discussions of mathematics, instead focusing on mathematically correct student contributions (Santagata, 2005; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005).

The purpose of this study is to examine how teachers' beliefs and knowledge influence their ways of handling students' errors during class discussion related to mathematics tasks. To support teacher transitions to reform-based mathematics programs and practices, better understanding of how teachers' beliefs and knowledge are related to specific reform-based teaching practices is needed (Ball, Lubienski, & Mewborn, 2001). Furthermore, understanding how teachers' beliefs and knowledge influence their handling of students' errors during class discussion seems especially important in the urban school setting, because the trend of lower academic achievement in urban schools suggests that teachers are likely to encounter a higher incidence of errors in these settings (Lippman, Burns, & McArthur, 1996). Focusing on teachers in their 1st year of using a reform-based mathematics curriculum in an urban school, this research was organized to explore the following questions: In what ways do teachers respond to student errors in the context of class discussion of mathematics? How do teachers' beliefs and knowledge influence their error-handling practices during class discussion of mathematics?

BACKGROUND OF THE STUDY

This study is informed by the growing body of research in mathematics education focused on determining best teaching and learning practices that support students in developing a strong conceptual understanding of mathematics. Reflecting constructivist views of learning, advocates of reform-based mathematics instruction believe that children actively construct increasingly organized structures of knowledge and personal understanding by reflecting on and reasoning about experiences in relation to their prior knowledge and immediate contexts (Carpenter & Lehrer, 1999; NCTM, 2000; von Glasersfeld, 1996). Therefore, students' differential realities and existing knowledge constructions become the starting point for conceptually supportive, reform-based instruction (Wood, Cobb, & Yackel, 1995).

Several research efforts have illuminated teaching strategies that make students' mathematical thinking a central feature of instruction (Fraivillig, Murphy, & Fuson, 1999; Franke, Fennema, & Carpenter, 1997; Hiebert et al., 1997; Kazemi & Stipek, 2001; Stigler, Fernandez, & Yoshida, 1996). Tasks that support student thinking are designed as opportunities for students to grapple with mathematics as problem solvers rather than as rule followers (Hiebert et al., 1997). These opportunities often encourage students to solve mathematics-rich tasks in their own ways (Franke et al., 1997; Stigler et al., 1996) with access to tools that support mathematical thinking (Hiebert et al., 1997). The social culture created in these classrooms is such that a variety of ideas and methods are valued, errors are treated as opportunities for learning, and correctness resides in the mathematical argument instead of with the teacher or text (Hiebert et al., 1997; Stipek et al., 1998).

A hallmark of reform-based mathematics pedagogy is small-group and whole-class discussions in which students and teacher work as a community to share, justify, compare, and interrogate multiple strategies for solving problems (Franke et al., 2007). In this *community of learners* approach, teachers are charged with facilitating discussion of students' approaches to mathematical tasks so that all students are actively involved in the discussion. Their role includes carefully inserting questions and explanations to ensure that the process and the conceptual underpinnings of students' mathematical strategies, as well as the focal mathematical ideas, are clear to all learners. Class participation associated with high levels of student thinking in such discussions involves students asking each other questions and making judgments about the strategies shared (Wood, Williams, & McNeal, 2006). Other research has highlighted the importance of having students explain each other's strategies and observe relationships among strategies (Fraivillig et al., 1999; Stein, Engle, Smith, & Hughes, 2008). Within this context, examination of students' flawed solutions during class discussion has been shown to be especially useful for facilitating insight into the mathematics of focus (Borasi, 1994; Kazemi & Stipek, 2001; Leinhardt & Steele, 2005).

As increasing numbers of teachers have attempted to adopt reform-based teaching practices that build on student thinking, it has become clear that orchestrating class discussion of mathematical ideas is quite challenging (Franke et al., 2007). Research has suggested that both teacher knowledge (Ball et al., 2001) and teacher beliefs (Knapp, 1995; Thompson, 1992) shape mathematics teaching practices. Whereas previous studies have examined how teacher knowledge and beliefs separately influence mathematics teaching, the aim of this study is to explore the relative influence of these variables on teachers' error-handling practices during class discussion of mathematics.

CONCEPTUAL FRAMEWORK

An interactive perspective of teachers' knowledge, beliefs, and experiences provides the underlying conceptual framework for this study. This perspective posits that teacher knowledge and beliefs are evolving cognitive constructs that are constantly reinforced or revised by the influence of experiences within and outside the classroom and each other. Additionally, teacher knowledge and beliefs are thought to be the lenses through which teachers interpret their experiences and decide which aspects of cognition are most important in a given situation. From an interactive perspective, a teacher's actions during mathematics instruction are simultaneously shaped by her knowledge and beliefs, with varying weight being given to particular types of knowledge or beliefs in different situations. Following is a brief discussion of the literature on teachers' knowledge and beliefs designed to offer insight into how these variables are thought to influence mathematics teaching.

Teacher Knowledge

Borko and Putnam (1996) identify three main categories of knowledge that support teaching: general pedagogical knowledge, subject matter knowledge, and pedagogical content knowledge. *General pedagogical knowledge* refers to important knowledge about teaching, learners, and learning that transcends particular subject matter domains. This includes knowledge about effective strategies for planning, classroom routines, conducting lessons, and classroom management, as well as general knowledge about how children think and learn. *Subject matter knowledge* refers to knowledge of the discipline of mathematics that is not unique to teaching. This includes knowledge of the important facts, concepts, and procedures, as well as knowledge of the concepts underlying the procedures and relationships between mathematical ideas.

Drawing on the work of Shulman (1986) and Grossman (1990), Borko and Putnam identified a third category of knowledge, pedagogical content knowledge, which refers to the unique set of subject matter knowledge used in teaching. As first described by Shulman (1986), *pedagogical content knowledge* includes, “the ways of representing and formulating the subject that make it comprehensible to others” and “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (p. 9). Grossman (1990) elaborated on Shulman’s initial description of pedagogical content knowledge to include knowledge of curriculum and curricular materials. Other researchers have highlighted the importance of understanding student cognitions in particular mathematical domains (Fennema & Franke, 1992).

Ball and colleagues have more recently elaborated the constructs of subject matter knowledge and pedagogical content knowledge in a framework of *mathematical knowledge for teaching* (Ball, Thames, & Phelps, 2008). This framework defines with greater specificity the boundaries between common content knowledge and specialized content knowledge for teaching mathematics. It also specifies types of pedagogical content knowledge, focusing on the intersections between knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum.

Schoenfeld (1998, 1999) and Leinhardt (1993) emphasize the importance of the ways teachers organize knowledge in mental schemas such as lesson images and action plans including routines and scripts. The primary function of a lesson image is to provide a conceptual roadmap charting the direction of a lesson (Schoenfeld, 1998, 1999). It includes the overarching goals and anticipated actions of a lesson and focuses primarily on the nonroutine parts of that lesson. In general, experts incorporate more detail into their lesson images than novices, and they have a better sense of how a given lesson will play out (Leinhardt, 1993). Routines are premeditated action plans that facilitate management and classroom norms (Leinhardt, 1993), and scripts are action plans for addressing content-specific issues (Schoenfeld, 1999). Expert teachers are more likely than novices to anticipate student misconceptions and common errors in advance of a lesson and to have scripts for addressing those issues (Borko & Livingston, 1989).

Teacher Beliefs

Teachers' beliefs are important because they influence teachers' perceptions and interpretations of events (Pajares, 1992) and serve as a guiding force in the kinds of actions teachers take (Cooney, Shealy, & Arvold, 1998). Teachers hold beliefs about subject matter, learners, and learning, as well as beliefs about teaching and the role of the teacher (Calderhead, 1996). These beliefs are thought to be held with varying degrees of conviction and consistency (Green, 1971). The complexity of teaching requires teachers to act in situations in which multiple, sometimes conflicting, beliefs are activated at once (Aguirre & Speer, 2000). The action a teacher chooses to take is thought to be, in part, a result of the prioritization of context-specific goals and strength of beliefs.

Within the mathematics education literature, there is a focus on the importance of teachers' beliefs related to the nature of mathematics, mathematics teaching, and mathematics learning (Franke et al., 1997; Thompson, 1992). According to Thompson (1992), "A teacher's conception of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (p. 132). Those with conceptions of mathematics most consistent with current reform efforts view mathematics as a dynamic discipline focused on solving problems by thinking creatively, finding patterns, and reasoning logically. Teachers' beliefs about the nature of mathematics have implications for how they will view and approach mathematics teaching (Lerman, 1983).

A teacher's beliefs about mathematics teaching include personal philosophies related to the most desirable goals of mathematics instruction, related instructional approaches and emphases, what counts as mathematical activity, and appropriate roles of teachers and students during classroom instruction (Thompson, 1992). Reflecting mathematics reform and its emphasis on student thinking, a *learner-focused* view concentrates on teaching in ways that support students' personal construction of mathematical knowledge. Consistent with a constructivist orientation to teaching, the teacher's role is to be a facilitator of student learning by asking probing questions and helping students uncover misunderstandings and new understandings (von Glasersfeld, 1995). In addition, Cooney and Shealy (1997) point out that teaching practices advocated by mathematics reform proponents require teachers to believe that multiple perspectives and partial solutions are valuable instructionally.

According to Fennema et al. (1996), teachers who are successful at building mathematics instruction around student thinking tend to hold certain beliefs about their students. They view children as coming to their classrooms with mathematical knowledge and the ability to acquire new knowledge by engaging in problem solving. Therefore, these teachers believe that students are capable of learning without direct instruction. In contrast, teachers in urban schools serving students from poverty have been found to be more likely to view instruction involving higher level thinking as inappropriate for their students, favoring instruction that is teacher-centered, highly controlled, and focused on basic skills (Knapp, Shields, & Turnbill, 1995; Spillane, 2001).

Even if teachers hold certain beliefs that are aligned with mathematics reform, it does not mean that they will necessarily act in ways consistent with those beliefs. Empson and Junk (2004) report a study in which teachers expressed beliefs that it is a good idea to use student mistakes as opportunities for learning. However, the actions described by several teachers in response to an instructional scenario did not reflect this professed belief. Empson and Junk suggest that lack of specific knowledge of children's mathematical thinking may limit teachers' abilities to act on beliefs. Additionally, Leatham (2006) suggests that teachers may act in ways that seemingly contradict espoused beliefs because their priority goal in a given moment of instruction is to accomplish something different from enacting the given belief.

In summary, teachers hold several types of knowledge and beliefs that are organized in ways that allow varying levels of influence on actions taken during mathematics instruction. In turn, these actions may facilitate or limit opportunities for students to develop conceptually grounded mathematical thinking. Further study of how and when teachers act on the knowledge and beliefs they hold is warranted. This study aims to tease out the relative influence of particular types of teacher knowledge and beliefs when student errors arise during class discussion of mathematics.

METHOD

A collective case study design (Stake, 2000) was employed to examine how urban elementary school teachers' beliefs and knowledge influence their error-handling practices during class discussion of mathematics. This method was selected because case study design is particularly suited to the study of complex phenomena in which the variables being studied cannot be separated from the context in which they exist (Yin, 2003). Using a constant comparative method to focus data collection and analysis (Glaser & Strauss, 1967), interpretive case studies on each of 4 third-grade teacher participants were developed using data collected over the course of 1 school year. A simultaneous cross-case analysis was undertaken to illuminate patterns across cases and increase the potential for generalizing beyond the particular cases (Merriam, 1998; Yin, 2003).

Setting

The site selected for this study was Lincoln Heights Elementary (a pseudonym), a school located in the southeast United States that exemplifies the challenges faced by many urban schools (Land & Legters, 2002). It is an identified Title I school in which the majority of students are living in poverty and are English-language learners. Table 1 presents demographic data for Lincoln Heights and the school district in which it is situated.

During the year of this study, teachers at Lincoln Heights were transitioning from use of a traditional mathematics text to the reform-based Everyday Mathematics program (Bell, Bell, & Hartfield, 1993) as a result of school-district mandates. In efforts to support this transition, Lincoln Heights' principal hired mathematics educators (including me) to provide a year-long program of professional development involving monthly grade-level-specific half-day meetings and opportunities to observe model teaching. I reasoned that a school in transition would offer an optimal site for this study because teachers were likely to have wide variation in the teacher knowledge and beliefs thought to be supportive of a reform-based mathematics program. At the same time, the sustained program of professional development would provide opportunities for teachers to understand the intent of the new reform-based mathematics program and develop knowledge for using it.

Case Study Participants

Merriam (1988) asserts that cases "should be selected for their power both to maximize and to minimize differences in the phenomenon of interest" (p. 154). Therefore, I set out to recruit teacher participants from a single grade to facilitate comparison across individual cases. I invited Lincoln Heights' third-grade teachers to participate in this study because our first professional development meeting suggested that teachers on this team varied with regard to years of teaching experience, mathematics-related beliefs, and comfort with mathematics. Of the 6 third-grade teachers, 4 agreed to be case study participants. The other 2 teachers declined participation due to the time commitment involved. Characteristics of the 4 case study participants are presented in Table 2. Pseudonyms are used to ensure the confidentiality of participants.

Of the 4 case study participants, only Ms. Aria reported having had positive personal experiences with school mathematics. Ms. Jarmin and Ms. Larsano identified themselves as "bad at" mathematics. The year of this study was the 1st year during which any of the case-study participants taught mathematics using a reform-based mathematics program.

Data Sources

Because this research focuses on how teachers' beliefs and knowledge influence their handling of student errors during class discussion of mathematics, analyses center on 16 classroom observations (4 per teacher) of lessons for which *Everyday Mathematics* directs teachers to facilitate class discussion of students' varied solution strategies for problems solved without explicit instruction. Two observations occurred as students were developing initial multiplication and division concepts in the fall. The other two observations occurred during instruction to extend ideas about multiplication to multidigit numbers in the spring.

Table 1
Composition of the Student Population at Lincoln Heights and Its School District

	Enrollment	Race/ethnicity (%)					School services (%)		
Group	*n*	Asian	Black	Hispanic	White	Other	Free/ reduced-price lunch	Special education	English language learners
Lincoln Heights	568	2	12	74	10	2	86	14	59
School district	170,000[a]	4	28	28	38	2	50	15	19

[a]Student enrollment at the school district level is intentionally approximated to protect the identity of the school district.

Table 2
Characteristics of Case Study Teacher Participants

	Personal characteristics			Education	Teaching experience	
Teacher	Gender	Race/ethnicity	First language	Highest degree	Years	Grades taught previously
Ms. Aria	Female	White	English	Bachelor's, Elem Ed	0	Grade 5 (intern)
Ms. Jarmin	Female	White	English	Bachelor's, Elem Ed	20+	Grades K–3
Ms. Larsano	Female	Hispanic	Spanish	Bachelor's, Elem Ed	10	Grades 1–6
Ms. Rosena	Female	Hispanic	Spanish	Master's, Elem Ed	7	Grades 4–5

During each lesson, extensive field notes were taken and audio recording was used to capture verbatim dialogue between each teacher and her students. These data sources were then integrated to form detailed observation transcripts of each lesson. Additionally, semistructured pre- and postobservation interviews were conducted to further illuminate teachers' instructional thinking before, during, and after these lessons (see select interview questions in Figure 1). General pre- and postinterview questions were modeled after items from *A Study Package for Examining and Tracking Changes in Teachers' Knowledge*, published by the National Center for Research on Teacher Learning (Kennedy, Ball, & McDiarmid, 1993, pp. 99–112). The postobservation interviews included a "lesson walk" in which field notes completed during the observation were used to reconstruct the

Preobservation Interview

1. Walk me through, step by step, what you are planning to do when I observe your class.
2. How do you anticipate your students will respond to ______? What strategies do you think they will use? What will be easy, and what will be difficult?
3. Is there anything in particular that you are hoping to have happen? How will that be accomplished?

Postobservation Interview

1. How do you feel things went during the observed lesson? How did things compare to what you expected? Did anything surprise you? Were there any ways that you (personally) felt challenged during this lesson?
2. Now I would like to walk through your lesson and ask questions about specific parts. (Use field notes to review lesson, inserting questions about particular aspects of the lesson.)

Typical Probes

- The selection of tasks/examples/representations
- Reasons for teacher moves during different parts of the lesson, especially related to student thinking
- Impressions of how students were thinking about the various tasks
- On-the-spot decision making

3. How did the urban school context influence your teaching of this lesson (if at all)?

Figure 1. Select questions from pre- and postobservation interviews.

lesson and provide a context for teachers to discuss instructional decisions made at various points.

To further explore teachers' beliefs and mathematical knowledge for teaching outside the classroom setting, the Integrating Mathematics and Pedagogy (IMAP) Web-based beliefs survey (Ambrose, Phillip, Chauvot, & Clement, 2003) and a teacher knowledge interview were completed at the beginning and end of the school year. Premeasures were used to establish a profile of teachers' beliefs and knowledge before they had significant interaction with the reform-based curriculum materials and professional development. Analysis of these premeasures informed professional development design and served as a tentative lens for classroom observations. It was expected that interaction with the reform-based curriculum materials and professional development would support increases in beliefs and knowledge aligned with reform-mathematics pedagogy, so parallel postmeasures were used to measure change about beliefs and knowledge generated through classroom observation data.

Belief 1: Mathematics is a web of interrelated concepts and procedures (and school mathematics should be, too).

Belief 2: One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.

Belief 3: Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.

Belief 4: If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely to ever learn the concepts.

Belief 5: Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect.

Belief 6: The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not.

Belief 7: During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.

Note. From *IMAP Web-Based Beliefs-Survey* (Integrating Mathematics and Pedagogy, n.d.).

Figure 2. Seven teacher beliefs measured by the IMAP survey.

The IMAP Web-based beliefs survey (Ambrose et al., 2003) is designed to measure teachers' adherence to seven beliefs (see Figure 2) aligned with reform-mathematics pedagogy using an interactive, Web-based platform in which teachers comment on a series of classroom-based scenarios presented through video and text. Rubrics that accompany the survey are designed to evaluate the degree to which teachers' responses provide evidence of each of the seven target beliefs. For example, one survey segment used to assess Belief 7 (During interactions related to the learning of mathematics, the teacher should allow the children to do as much thinking as possible) presented a video of a teacher providing step-by-step guidance on how to solve a contextualized division problem. Following the video, survey respondents were first asked to share what "stood out" to them in the clip of instruction. After submitting a response, they were asked to identify strengths and weaknesses of the teaching in the episode. Finally, respondents were asked explicitly whether they thought the child could have solved the problem with less help. The rubric used to assess Belief 7 focused on the extent to which there was evidence that the respondent found the video teacher's guidance to be excessive and the strength of that evidence. A respondent who commented on the excessive teacher guidance when asked to give general impressions on the first prompt was considered to provide stronger evidence of Belief 7 than a respondent who brought this up in the subsequent prompts.

The teacher knowledge interview was a semistructured interview in which teachers responded to four specific teaching scenarios (see Figure 3). Based on scenarios developed by Kennedy et al. (1993) and Empson and Junk (2004), these scenarios asked teachers to anticipate the varied ways children might solve a particular problem, interpret student work, and respond to students' difficulties. The interviews were audio recorded and transcribed for use in analyses.

Lastly, I maintained a research log of informal classroom observations, conversations with case study teachers, and teacher development project meetings. This log served as an additional data source and included a record of the contents of observations, conversations, and meetings as well as my reflective thoughts, impressions, and developing conjectures.

Data Analysis

In keeping with qualitative research design (Lincoln & Guba, 1985), I began analyzing data as soon as I collected them. The beginning-of-year measures of beliefs and knowledge and the data from early professional development meetings provided an initial understanding of each teacher's beliefs and knowledge. These early understandings served as a lens that shaped my subsequent interviews with and observations of each teacher. As conjectures about the nature of the influence of teachers' beliefs and knowledge on instructional practice were formulated, I continuously sought evidence to confirm or disconfirm these conjectures. In particular, the lesson-walk portion of the postobservation interview provided opportunity for me to ask teachers directly to explain their instructional thinking

Classroom Scenario 1: Knowledge of Nonstandard Strategies

Teachers brainstorm, record, and discuss multiple strategies students might use to solve a multidigit multiplication problem involving finding the number of chairs in 16 rows with 8 chairs per row.

Classroom Scenario 2: Interpretation of Students' Mathematical Strategies[a]

Teachers are presented with three in-progress student work samples for a partitive division problem in which 24 children are sharing 8 pancakes. Teachers are prompted to describe how each student appears to be approaching the problem and to discuss what each strategy suggests about the student's mathematical understanding. Teachers are also asked to identify questions they would ask each student to probe and extend his or her thinking.

Classroom Scenario 3: Addressing and Avoiding a Common Student Error[b]

Teachers are presented with a sixth-grade work sample in which the standard U.S. multiplication algorithm is executed for the problem 645×123 without maintaining the place values of the partial products. Teachers are prompted to discuss how they would address this error with sixth-grade students and how instruction in third-grade can be designed so that students are less likely to make this kind of error.

Classroom Scenario 4: Interpretation of and Response to a Flawed Solution[c]

Teachers are presented with a student's work and an explanation of an unusual flawed solution to the division problem $144 \div 8$. Teachers are prompted to explain the mistake and what it suggests about the student's mathematical understanding. Then, teachers describe how they would respond to the student instructionally.

[a]The pancake problem and student work used for classroom scenario 2 are from Empson (2001).
[b]Classroom scenario 3 is an interview item from Kennedy, Ball, and McDiarmid (1993).
[c]Classroom scenario 4 is an interview item from Empson and Junk (2004).

Figure 3. Summary of knowledge interview protocol.

in relation to specific actions within observed lessons. This, in turn, led to revision of conjectures and development of new conjectures. Additionally, themes that emerged through study of one teacher were intentionally explored in my observations of and interviews with the other teachers. Although I experienced the relationships among analyses of different data sources as recursive and cumulative, for the sake of clarity in the following paragraphs, I discuss component analyses as discrete and sequential events.

To analyze teachers' responses on the IMAP Web-based beliefs survey, I used the set of rubrics that accompany the survey as directed to evaluate seven target beliefs considered central to reform-oriented mathematics teaching (Integrating Mathematics and Pedagogy, n.d.). The beliefs rubrics differentiate evidence of a teacher's beliefs using a scale ranging from 0 (no evidence of belief) to 3 (strong evidence of belief). Beginning- and end-of-year responses to the teacher-knowledge interview were analyzed in relation to the research literature on knowledge that has been theorized to be supportive of teaching multiplication and division concepts in a reform-oriented manner. Specifically, the following four facets of teacher knowledge relating to the teaching and learning of multiplication and division concepts were explored:

- Knowledge of key mathematical concepts,
- Knowledge of student strategies,
- Knowledge of teaching strategies shown to support development of conceptual understanding, and
- Ability to use mathematical knowledge to interpret student work.

First, interviews were coded according to these different facets of teacher knowledge. Then, at each data point, a summary of teacher knowledge suggested by each interview was constructed.

To understand each teacher's error-handling practices in the context of class discussion of mathematics, I first reviewed observation transcripts to identify action sequences during discussions in which the teachers encountered and responded to (or did not respond to) real or perceived student errors. Coded action sequences included the evidence of the particular student error, the teacher's response, and any text needed to contextualize the teacher's response. In addition, memos were attached with additional relevant information. Next, the constant comparative method (Glaser & Strauss, 1967) was employed to characterize each teacher's typical patterns of response to student errors during class discussion of mathematics tasks.

After each teacher's error-handling practices were established, data from the pre- and postobservation interviews and beginning- and end-of-year measures of teachers' beliefs and knowledge were used to analyze each teacher's response patterns in relation to her beliefs and knowledge. In addition to attending to practices that remained relatively stable, I looked for patterns of change in practice that corresponded to patterns of change in particular aspects of beliefs and knowledge.

These patterns of change were tracked by first examining practice-based evidence of beliefs and knowledge in a given observation linked to relevant teacher moves and the comments teachers provided in pre- and postobservation interviews that were associated with those moves. I then identified whether evidence of particular beliefs and facets of knowledge from a given observation more closely matched beginning- or end-of-year measures of beliefs and knowledge. Finally, I looked across observations for a given teacher to determine whether patterns of change appeared to move in one direction or if they were more variable and situation-specific.

As the relationship between each teacher's actions and her beliefs and knowledge emerged, records of informal observations and professional development meetings were searched for participants' explanations and actions that contradicted or confirmed identified response patterns, beliefs, and knowledge. The outcome of these analyses was the construction of case stories offering a theoretical explanation of how each teacher's ways of responding to student errors during class discussions were related to her beliefs and knowledge.

Throughout data analysis, themes that emerged in one teacher's instruction were explored across the other cases. After interpretive case stories were developed for each teacher, these case stories were reviewed with the purpose of identifying dimensions of teachers' error-handling practices during class discussion of mathematics tasks that served to capture the response patterns of the 4 teachers studied. Next, the theoretical explanations for how beliefs and knowledge influenced an individual teacher's actions were examined in relation to each dimension of teacher response. Through this analysis, a theoretical explanation was developed for how teacher beliefs and knowledge appear to influence particular teaching practices related to error-handling. Through this process, aspects of teacher actions that appear to be most influenced by teacher beliefs and those most influenced by teacher knowledge were identified.

Establishing Trustworthiness

Throughout the research process, steps were taken to strengthen the trustworthiness of the research findings based on Guba and Lincoln's (1989) framework for establishing trustworthiness. This study was designed to allow for prolonged engagement and persistent observation in the research setting, as well as triangulation of data from multiple sources. Formal interviews and informal conversations with teachers provided a venue for member checks designed to test and refine initial interpretations of data. Peer debriefing with a veteran researcher with expertise in qualitative methods (she has taught classes on qualitative methods) was used to interrogate the basis of developing conjectures about teachers' patterns of error-handling practices and theoretical claims regarding how these practices related to the teachers' beliefs and knowledge. For a given claim, my peer debriefer first had me identify confirming and disconfirming evidence from the data. She then probed possible explanations for the disconfirming evidence and, at times, I revised the claim based on the weight of evidence. An audit trail of audio tapes, transcripts, researcher notes, and memos was maintained and referenced over the course of this research. Through this process of checks, I sought to establish findings that were credible, dependable, and confirmable.

FINDINGS: TEACHER BELIEFS AND KNOWLEDGE MEASURES

This study examines how teachers' beliefs and knowledge influence their error-handling practices during class discussion of mathematics. Before findings on teachers' error-handling practices are elaborated, results from beginning- and end-of-year measures of the teachers' beliefs and knowledge are presented to provide a foundational understanding of these variables across participants.

IMAP Web-Based Beliefs Survey Results

A summary of the IMAP Web-based beliefs survey results for the 4 teachers studied is presented in Table 3. Results suggest that none of the 4 teachers studied had a beliefs profile aligned with reform-oriented mathematics teaching at the beginning of the year. Ms. Aria's beginning-of-year profile came closest to a reform-orientation, with IMAP results suggesting strong evidence of three of the seven beliefs assessed. Ms. Jarmin and Ms. Rosena both held beliefs profiles at the beginning of the year in which there was weak or moderate evidence of six of the seven beliefs measured and no evidence of one belief. Ms. Larsano's beliefs profile at the beginning of the year was the farthest from a reform orientation, with no evidence of four beliefs and weak evidence of the other three. Particularly notable was the fact that there was weak or no evidence at the beginning of the year that any of the teachers believed children capable of solving mathematics problems without being taught methods (Belief 5) or that teacher–student interactions should allow students to do as much of the thinking as possible (Belief 7).

At the end of the year, there was evidence that three of the four teachers' beliefs made significant shifts toward a beliefs profile aligned with reform-oriented mathematics instruction. There was strong evidence that Ms. Aria held six of the seven beliefs measured by the IMAP at the end of the year, with moderate evidence of the seventh belief. For Ms. Rosena, there was also strong or moderate evidence of all seven beliefs. For Ms. Aria and Ms. Rosena, the dramatic changes in responses to items associated with IMAP Beliefs 5 and 7 are notable. By the end of the year, these teachers appeared to believe that children are capable of solving novel mathematics problems without being shown particular methods, and they seemed to believe that interactions with children during mathematics instruction should encourage students to do as much thinking as possible. Ms. Jarmin's responses on the IMAP survey at the end of the year suggested a greater degree of evidence for five of the seven reform-oriented beliefs measured, but strong evidence for only two of the seven beliefs. Finally, Ms. Larsano's IMAP survey results suggested little change in beliefs toward a reform orientation, with weak or no evidence that Ms. Larsano held six of the seven beliefs measured at the end of the year.

Table 3

Summary of IMAP Web-Based Beliefs Survey Results by Teacher

Beliefs	Ms. Aria		Ms. Jarmin		Ms. Larsano		Ms. Rosena	
	Beg.	End	Beg.	End	Beg.	End	Beg.	End
Belief 1: Mathematics is a web of interrelated concepts and procedures (and school mathematics should be, too).	2	3	2	3	1	1	1	1[a]
Belief 2: One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.	3	3	1	2	0	0	1	2
Belief 3: Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.	3	3	2	3	0	1	2	3
Belief 4: If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely to ever learn the concepts.	3	3	2	2	1	2	1	2[a]
Belief 5: Children can solve problems in novel ways before being taught how to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expect.	1	3	0	2	0	1	1	3
Belief 6: The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking, whereas symbols do not.	1	2	1	1	1	1	2	2
Belief 7: During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.	1	3	1	2	0	0	0	3

Note. 0 = No evidence; 1 = Weak evidence; 2 = Moderate evidence; and 3 = Strong evidence.

[a]Evidence from classroom observations suggests that the end-of-year ratings underestimate Ms. Rosena's adherence to beliefs 1 and 4. Classroom observations provide quite strong evidence of these beliefs.

Knowledge Interview Results

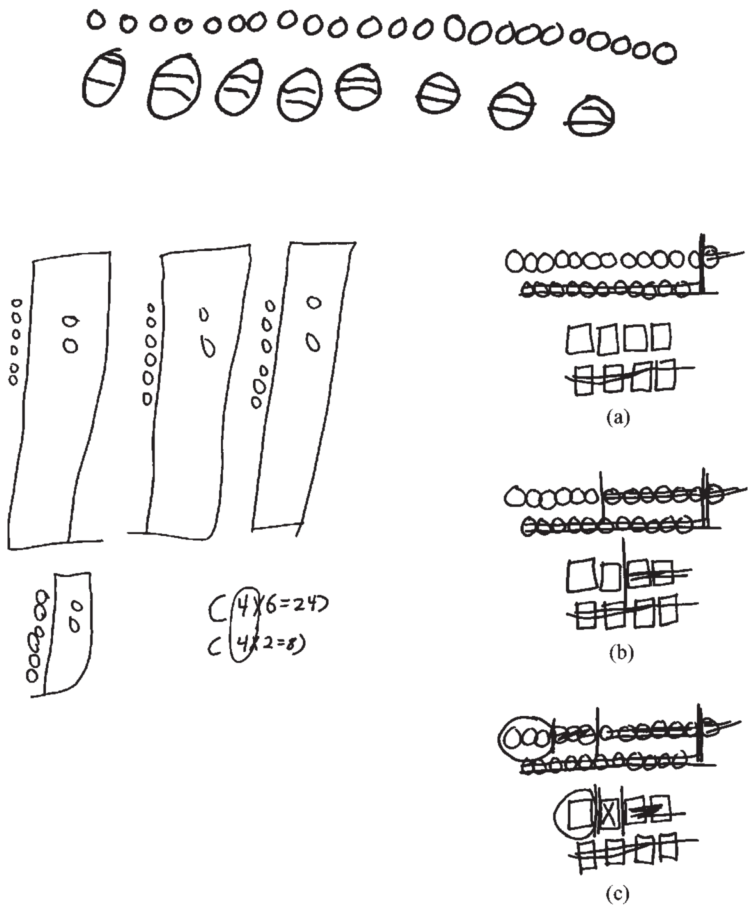
The knowledge interview, conducted at the beginning and end of the year, used four classroom scenarios related to multiplication and division instruction to probe teachers' knowledge of key mathematical concepts, student strategies, and conceptually supportive teaching, as well as teachers' abilities to use mathematical knowledge to interpret student work.

Classroom scenario 1 prompted teachers to brainstorm and discuss multiple strategies that students might use to find the total number of chairs in 16 rows of 8 chairs. At both data points, all teachers identified one or more direct modeling strategies, one or more addition-based strategies (i.e., skip counting or repeated addition), and the standard U.S. algorithm. A multiplication-based strategy of partitioning by place values (i.e., $10 \times 8 + 6 \times 8$) was identified by Ms. Aria and Ms. Jarmin at both data points and by Ms. Rosena at the end of the year. Despite the emphasis on the strategy of partitioning by place values in *Everyday Mathematics*, Ms. Larsano did not identify it—at either data point—as a strategy that students might use. However, she did share a partitioning strategy of breaking 8×16 into $4 \times 16 + 4 \times 16$ at the end of the year. In addition, Ms. Aria and Ms. Rosena each described two additional multiplication-based strategies at the end of the year, demonstrating flexible understanding of how the distributive property can be used in different ways to solve multidigit multiplication problems.

Classroom scenario 2 asked teachers to describe how three students appeared to be approaching a partitive division problem resulting in a fraction, given the in-progress student work samples shown in Figure 4. Teachers were prompted to explain what each strategy suggested about the student's mathematical understanding and to identify questions they might ask to probe and extend student thinking. After examining student work at both data points, Ms. Aria and Ms. Rosena quickly and fluently described plausible theories for how each student approached the task, indicated what each approach suggested about mathematical understanding of key mathematical ideas, and identified conceptually supportive follow-up questions. Ms. Jarmin struggled with making sense of student work presented in classroom scenario 2 at both data points, requiring significant researcher support and time to interpret all three samples at the beginning of the year and one sample at the end of the year. Ms. Larsano provided reasonable explanations of two of the three student work samples at both data points. When Ms. Larsano did not understand a strategy, at both data points she indicated that she thought the student was probably guessing and that she would direct them to use a different strategy. At the end of the year, follow-up questions identified by Ms. Jarmin and Ms. Larsano for strategies they understood demonstrated greater intent to have students justify their strategies.

In classroom scenario 3, teachers were presented with a sixth-grade student work sample in which the standard U.S. multiplication algorithm was executed for 645×123 without maintaining the place values of the partial products. After identifying the error, each teacher was asked to discuss how she would address this

Problem: Twenty-four children want to share eight pancakes so that each one gets the same amount. How much pancake can each child have?



Note. The problem and work samples are from "Equal Sharing and the Roots of Fraction Equivalence," by Susan B. Empson, 2001, *Teaching Children Mathematics, 7*, 421–425. Copyright © 2001 by the National Council of Teachers of Mathematics. Reprinted by permission. All rights reserved.

Figure 4. Problem and student work samples used in classroom scenario 2 of the knowledge interview.

error with sixth-grade students and how instruction in third grade can be designed to avoid this error. Ms. Larsano's responses at both data points indicated that she knew place value was involved in the error, but she was not entirely clear how. At both data points, Ms. Larsano indicated that she would respond to students making the error by having them work problems on grid paper or draw lines to emphasize where they should put numbers. The other three teachers (Ms. Aria, Ms. Jarmin, and Ms. Larsano) all demonstrated strong understanding of the relationship between place value and the error in the multiplication problem. In describing how she would address the error or help students to avoid the error, Ms. Jarmin described a teacher-centered and somewhat procedural approach at both data points:

> I would just explain over and over and over that the 6 doesn't stand for a 6 and the 4 doesn't stand for a 4. So, you know that the 4 stands for 40, so you always know, whatever you are multiplying by in the tens position, it is going to end in one zero.

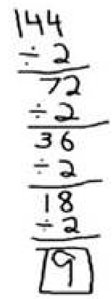
At the end of the year, Ms. Jarmin also shared that it would be helpful to show students a model of the partial products with manipulatives but said that she was unsure how to do that. At both data points, Ms. Aria and Ms. Rosena described conceptual approaches to addressing the error with sixth-graders by having them focus on understanding what each place represents through a process of breaking down the problem into subproblems (i.e., partial products). At the end of the year, Ms. Aria, Ms. Jarmin, and Ms. Rosena all stressed the importance of infusing place value language into the study of all multidigit operations with third-grade students and using conceptually explicit algorithms. Ms. Aria also described how making magnitude estimates is a strategy that supports avoiding this error.

Classroom scenario 4 prompted teachers to interpret and respond to the flawed nonstandard division strategy presented in Figure 5. At both data points, Ms. Aria was the only teacher able to make sense of the student's error. She indicated that she would respond to the student by guiding her to recognize the error by modeling the (flawed) strategy with manipulatives and emphasizing the link between the actions in the strategy and division concepts. At both data points, Ms. Jarmin and Ms. Larsano reported that they would respond to this student by directing her to try a different strategy. At the beginning of the year, Ms. Rosena said that she was unsure of how she would respond. At the end of the year, Ms. Rosena asserted that she would have the student compare her work to the work of her peers and use the class community to figure out why her strategy was not working.

Summary of Findings From Beliefs and Knowledge Measures

In summary, the IMAP Web-based beliefs survey results indicated that none of the 4 teachers studied held consistently strong reform-oriented beliefs at the beginning of the school year. End-of-year survey results suggest that all teachers except Ms. Larsano made significant shifts toward reform-oriented beliefs during the school year. Results from the knowledge interview revealed that Ms. Aria and Ms. Rosena demonstrated strong conceptual knowledge of mathematical concepts at both data points, with comparatively stronger knowledge of student thinking and

A student was solving 144 ÷ 8 (show card). She said, "I know, I can just split it in half. So I will keep dividing by 2. I need to do that 4 times, since 2 + 2 + 2 + 2 = 8. As she talked she wrote:



How would you respond to this student?

Note. The problem and work samples presented are from "Teachers Knowledge of Children's Mathematics After Implementing a Student-Centered Curriculum," by Susan B. Empson and Debra L. Junk, 2004, *Journal of Mathematics Teacher Education, 7*, Table 1, pp. 126–127. Reprinted with kind permission from Springer Science+Business Media.

Figure 5. Problem and student work samples used in classroom scenario 4 of the knowledge interview.

conceptual teaching strategies at the end of the year. Although Ms. Jarmin and Ms. Larsano also demonstrated growth over the year, evidence suggests that their knowledge for teaching mathematical concepts was comparatively weaker. These findings from beginning- and end-of-year beliefs and knowledge measures were consistent with trends found in other data sources, including formal classroom observations and data from informal observations and professional development meetings.

FINDINGS: TEACHERS' ERROR-HANDLING PRACTICES

To illustrate the variation in teachers' error-handling practices during class discussion of mathematics and its relationship to teachers' beliefs and knowledge, excerpts from case stories of 2 of the 4 teachers are shared.[1] These excerpts were selected because they contrast significantly, illuminating the range of error-handling practices observed among the 4 teachers studied. For each case story excerpt, aspects of the teacher's pattern of response to student errors in class discussion are depicted through a representative classroom episode. This is followed by discussion of the observed relationship between the response pattern and student learning. Finally, evidence of the teacher's beliefs and knowledge thought

[1] See Bray (2008) for complete case stories of all 4 teachers.

to contribute to the response pattern depicted in the episode will be detailed. To protect the confidentiality of students, all student names have been replaced with pseudonyms.

Case Story Excerpt—Ms. Rosena

Of the 4 teachers studied, Ms. Rosena was the only teacher who reported intentional incorporation of flawed solutions during class discussion of mathematics. The classroom episode that follows illustrates Ms. Rosena's typical practice with her class of retained students, all of whom were repeating third grade due to nonpassing scores on the state reading test in the previous school year. Although scores on the state mathematics test were not used in promotion decisions, it is important to note that none of Ms. Rosena's students received a passing score on that test either. This episode took place during a fall semester lesson introducing the array representation of multiplication. After some exploration, Ms. Rosena initiated a lengthy class discussion addressing solutions to the following task:

> There are 24 children in the class. Use the counters to represent the children. Arrange the counters to show them in equal rows.

Prior to class discussion, each student devised a solution with counters and represented it by drawing on a whiteboard. Then Ms. Rosena directed students to exchange whiteboard solutions with a partner and to try to understand how the partner was thinking about the problem. After a few moments, Ms. Rosena said, "Thumbs up if you understand. I'm not saying if it's right or wrong, if you understand what the person did, thumbs up." Although a few students indicated that they understood their partner's solution, several students indicated that they were not sure what their partner was thinking. Some of the students commented that their partner's solution was incorrect. At this point, several solutions were discussed by the whole class, with ample attention given to those solutions that were incorrect.

The episode that follows presents the whole-class discussion of Jeremy's solution, which was one of three flawed solutions discussed. Jeremy's whiteboard contained 2 columns of dots, with 14 dots in the first column and 10 dots in the second column. The dots were placed such that the columns were lined up, but the rows were not clear (see Figure 6). The discussion of Jeremy's solution began with Kamal, the recipient of Jeremy's whiteboard, talking about his understanding of Jeremy's work.

Ms. Rosena:	Kamal, you did not understand what Jeremy did?
Kamal:	Yeah.
Ms. Rosena:	Okay. Can you tell me or ask him to explain to the class what he did.
Kamal:	Because he didn't have an equal row . . .
Ms. Rosena:	He didn't have equal rows?
A few students:	Yeah; no.
Ms. Rosena:	Where do you see . . . why do you say that he doesn't have equal rows? I want Christy and Tanya to pay attention to this.

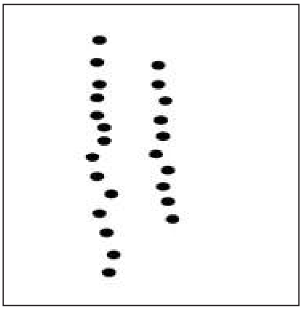


Figure 6. Jeremy's work.

Kamal: The first one, he has I don't know how much. But the second one he has . . .

Ms. Rosena: Okay. Explain that once more.

Kamal: The first one, I don't know how much he got in it.

Ms. Rosena: Yeah, me neither. Because the first one is up here and the second one down here. So, in the first one [pointing at the first row] there is only one, I think. [Pause] Jeremy, can you explain to the class what you did here. But, before you do that, are there 24 dots in here?

Kamal: Yeah.

Ms. Rosena: So, he does have 24. Jeremy, can you explain to the class how you came up with this arrangement.

Jeremy: Well, I put 1 row, I mean 1 column of 14 dots and another column of 10.

Ms. Rosena: Okay, so this 14? [Indicates the first column, Jeremy nods] And this is 10? [Indicates the second column, Jeremy nods] And that is 24. [Pause] But my question was to show the 24 children in equal rows. What does this mean to you—equal rows? What does this mean?

Jeremy: Like, if you have 2 [other children chiming in answers].

Ms. Rosena: Shhh.

Jeremy: If you have 2, you need to put 2 blacks on the other side. [Other students are agreeing.]

Ms. Rosena: If you have 2 on this row [indicates first row], you must have 2 on the second row, you must have 2 on the next row, etcetera. Did you do that?

Jeremy: No.

Ms. Rosena: Okay.

Jeremy: But if I have 14 and 14, that would be . . . 28.

Ms. Rosena: If you have 14 and 14, that would be 28. Okay. Is there a way that you can arrange this to make 24? To have equal rows making 24 in total?

At this point, Ms. Rosena directed the class to think about how many rows of 2 children make 24 children total. One student tentatively suggested 12 rows. Ms. Rosena replied, "Let's see," and guided the class to count by twos as she drew a model on the board. After the class was convinced that 24 children could be arranged in 12 rows of 2, Ms. Rosena remarked, "So Jeremy, if you wanted to do 2 in each row, you needed to have 12 rows."

In this episode, Ms. Rosena first had Kamal try to make sense of Jeremy's work, thus giving the class the opportunity to study the work too. Then Ms. Rosena guided Kamal and the rest of the class to understand the correct aspects of Jeremy's flawed solution (that there are 24 children represented) as well as the error (that they are not placed in equal rows). Next, Ms. Rosena probed Jeremy's understanding of the concept *equal* as it related to this situation. Through doing this, Ms. Rosena emphasized the mathematical idea of equal groups that is central to understanding multiplication and division. Finally, Ms. Rosena showed Jeremy, with the help of the class, how to formulate a correct solution using pieces of Jeremy's initial solution. Namely, they worked to figure out how many rows of two were needed to make 24.

Consequences for student learning. Ms. Rosena's primary response pattern for responding to students' errors involved having students analyze and revise flawed solutions during whole-group discussion. By midyear, Ms. Rosena's students appeared to view errors as a part of the learning process. They did not seem uncomfortable or unhappy when their flawed solutions were shared. Quite the opposite, students appeared motivated to understand correct as well as flawed solutions. Especially in the latter part of the year, there was a notable sense of collaboration among the students. It was not uncommon for students to comment on another student's solution strategy without being prompted by the teacher. Additionally, students were observed making significantly fewer errors in spring mathematics lessons than in fall mathematics lessons. Ms. Rosena's practice of making students' errors a focus of instruction appears to have been a contributing factor to this positive change.

Link to beliefs and knowledge. Ms. Rosena's intentional practice of making students' flawed solutions a focus of whole-class discussion seems most related to her beliefs but also supported by her knowledge. At the end of the year, Ms. Rosena explained that this very deliberate teaching practice stems from her own school experience:

> When I was in school, I was not good in math. So, the teachers were always showing off the people who did the right thing and were right. But they never, for example, they would never take my way of solving the problem and explain to me why it was wrong. So, I never really quite got how to do it right. So, I think that having them share how did they come up with an answer, even when it is wrong . . . I think that makes it so I can help them understand what it is they were doing wrong, so they can make it right in the future. I think that we are focusing on what students are doing right and sometimes not enough on what they are doing wrong.

Ms. Rosena expressed a deep belief that students' errors need to be confronted directly and, after the errors are identified, that they should be revised into correct solutions. Postobservation interviews revealed that Ms. Rosena focused on particular flawed solutions in the whole-group discussion because she believed the mathematical lesson of the error was beneficial for the given student and the rest of the class. She espoused the view that there was much for her students to learn from their own mistakes as well as from those of their peers.

Contributing to her belief in the value of studying errors, there is evidence that Ms. Rosena viewed students' flawed solutions as partially correct solutions with underlying logic. This is in contrast to a view of flawed solutions as the result of careless mistakes or unfounded guesses. This belief drove Ms. Rosena to examine students' incorrect solution strategies to understand where they went wrong, and, as in the case of Jeremy, whether the error was grounded in a misunderstanding of a fundamental mathematics concept. Ms. Rosena's end-of-year IMAP beliefs survey responses provided increased evidence of multiple beliefs related to conceptual understanding. In practice, Ms. Rosena placed increasing emphasis over the course of the year on having students explain the conceptual underpinnings of solutions focused on during class discussion.

As the year progressed, there also was evidence that Ms. Rosena developed stronger adherence to a belief in her students' capabilities to contribute productive ideas to each other during class discussion of mathematics. She became intentional in her efforts to structure student–student interaction, as evidenced by her strategy of having students exchange and comment on each others' solutions in the previous episode involving Jeremy's solution. Although Ms. Rosena made significant progress in orchestrating student participation during class discussion, shifting discussion to incorporate greater student talk and less teacher talk was an area of struggle for her throughout the year.

This exemplifies that, although Ms. Rosena's beliefs guided her to emphasize discussion of student errors, her knowledge for teaching mathematics played a significant role in shaping the details of her instructional response. In general, Ms. Rosena's conceptually supportive response to student errors during class discussion relied on her ability to make sense of student solutions, knowledge of key mathematical ideas related to multiplication and division, and pedagogical content knowledge of how to ask pertinent questions to probe and extend students' mathematical thinking. However, Ms. Rosena was still working on developing routines to promote greater productive student–student interaction during class discussions.

Case Story Excerpt—Ms. Larsano

In contrast to Ms. Rosena's approach, Ms. Larsano typically initiated class discussion of students' solutions with only a vague idea of the solutions her class of English Language Learners (ELL) had produced. She primarily relied on volunteers to share their problem solutions while also aiming to select students who, at a glance, appeared to have different strategies for a given problem. Although Ms. Larsano never reported intentionally having a student share a flawed solution, multiple flawed solutions were presented during whole class discussion in every lesson observed. In three out of four formal observations, flawed solutions shared publicly were not addressed at all by Ms. Larsano, because she reported not to have noticed the errors. Although this finding is disturbing, the episode that follows depicts one response pattern for Ms. Larsano's handling of errors identified during instruction.

The following episode is from a fall lesson in which Ms. Larsano was leading the class in discussing the problem, "23 candles are arranged with 3 in each row. How many rows are there?" Five students put their solutions to the candle problem on the board, and Ms. Larsano identified that Andre's solution (see Figure 7) did not match the problem context:

Ms. Larsano: Let's look at this one [Andre's solution] over here. Look at what we have here. We have [pointing at each column] 8, 8, 8. What's the total?

A few students: Twenty-four.

Ms. Larsano: What's the total?

Several students: Twenty-four.

Ms. Larsano: Twenty-four. But I started out with how many?

A few students: Twenty-three.

Ms. Larsano: Twenty-three. So, I have how many extra?

A few students: One.

Ms. Larsano: One extra. So, I have to take one from here [erases the bottom right circle from Andre's model, as in Figure 8].

Ms. Larsano: And now it wouldn't be equal, it wouldn't be equal. We have to change it, because the problem says, "Twenty-three candles, 3 in each row." And I have how many? [Pointing at each column] I have 8, I have 8, and I have 7. [Pause] You're looking, you're looking, you're seeing it? [To Andre] I'm not telling you that you are wrong, I am just explaining to you what you did. So, he put that 24 divided by 8 equals 3. So, he did have 3 groups. The only problem was that he put one more. But it's okay, it's a model.

In this instructional episode, Ms. Larsano first asked the class a series of closed questions to establish that Andre's model contains 24 candles instead of the 23 candles specified in the problem. This exchange was highly controlled by Ms. Larsano and did not invite Andre's peers to evaluate his solution. Then, Ms. Larsano revised the model by erasing one of the objects and pointing out that, with only 23 objects, the columns were not equal. Up to this point, Ms. Larsano's actions indicated that having 24 objects represented did not match the problem. But, at this point, she turned to Andre and said, "I am not telling you that you are wrong," which

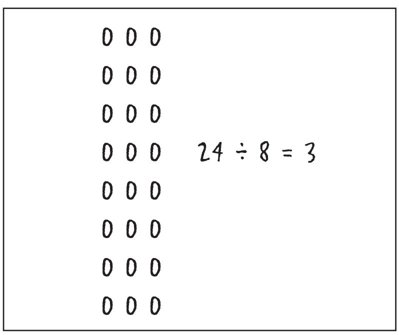


Figure 7. Andre's work on the board.

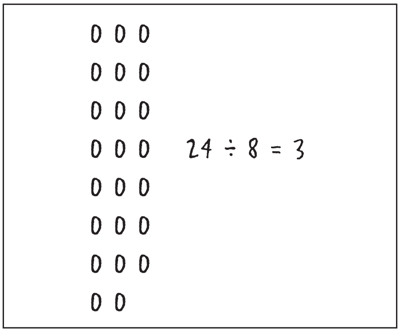


Figure 8. Andre's work on the board, as modified by Ms. Larsano.

contradicted previous actions and introduced uncertainty regarding whether Andre's approach was correct. Next Ms. Larsano addressed Andre's number sentence, $24 \div 3 = 8$, and identified that he had three groups. This was confusing because the 3 in the problem denoted the number of candles in each row, not the number of groups. Before moving on to another student's work, Ms. Larsano commented to the class, "But it's okay, it's a model," suggesting that if one makes a model of some kind, it might be considered a correct answer. The correct and incorrect aspects of Andre's model and number sentence were never made clear. Although the revised model could be used to determine the solution to the rows of candles problem, that was not brought out. Furthermore, at the end of the episode, Andre's number sentence, which remained on the board, matched neither the problem context nor the revised model.

Consequences for student learning. When Ms. Larsano noticed flawed solutions that surfaced in whole-class discussion, her intention was to guide the class to recognize and correct the error. However, this intention was often not realized, because Ms. Larsano maintained tight control over discourse and sometimes provided explanations that were conceptually unsupportive and fell short of making correct and incorrect aspects of flawed solutions clear. Furthermore, because Ms. Larsano prompted student participation primarily using closed questions, there was limited opportunity for students' ideas pertaining to flawed solutions to enter into the discourse. Ms. Larsano's reluctance to tell students publicly that their solutions were incorrect, coupled with the incidence of incorrect solutions going unaddressed, appeared to contribute to ambiguity regarding mathematical correctness. Furthermore, on occasions when Ms. Larsano lost track of the logic of a solution and shared mathematically confusing or incorrect information, students rarely questioned her flawed or incorrect assertions. Instead, they appeared to follow their teacher's presentation, recording flawed solutions in their notebooks and responding to the closed questions posed. Rather than thinking critically about the mathematical ideas put forward during discussions, Ms. Larsano's students appeared to accept as true the information stated by their teacher or displayed on the board.

Link to beliefs and knowledge. Ms. Larsano's handling of student errors during class discussion seems highly related to her knowledge and beliefs. Throughout the year, Ms. Larsano expressed that she felt challenged and sometimes overwhelmed by the teaching demands of *Everyday Mathematics* lessons, especially during whole-class discussions. Ms. Larsano consistently had difficulty making sense of students' strategies and errors both in-the-moment during instruction and in interviews and professional development meetings away from the pressures of real-time teaching. Ms. Larsano's response to Andre's flawed solution was intended to help students see how the response was flawed. However, she did not unpack Andre's solution in such a way that correct and incorrect aspects of the solution were illuminated. Furthermore, observation and interview data suggest that Ms. Larsano's pedagogical knowledge for facilitating student discussion and debate, during mathematics time or otherwise, was limited. In the episode focused on Andre's solution, Ms. Larsano dominated the discourse, limiting student participation to responding to closed questions.

Ms. Larsano's beliefs also contributed to her response pattern. First, consider Ms. Larsano's practice of avoiding telling students that their answers were incorrect in the public forum of whole-class discussion. Underlying this intentional teaching practice were several beliefs held by Ms. Larsano including beliefs that (a) incorrect answers would hurt students' feelings and discourage them from participating in the future; (b) ELL students are fragile and unable to handle being corrected in a public forum or to do much to support each other's learning from flawed solutions; and (c) ELL students develop language when they share ideas in whole-group discussions, so having a classroom culture that positively reinforces student participation is very important. Additionally, Ms. Larsano's practice of maintaining tight control of class discussions when flawed solutions arose was, in part, related to a belief that her students were limited in their capabilities to understand mathematics problems before they had been explicitly taught procedures for solving them. Consequently, Ms. Larsano believed that her role as their teacher was to provide necessary explanation and direction on how to solve problems. All these factors appear to have contributed to her practice of dominating the discourse during class discussion when student errors surfaced and then to the practice of not clearly identifying the correct solutions to problems shared as a whole group in a conceptually supportive way before moving on.

DISCUSSION ACROSS CASES

Through cross-case examination of all 4 teachers, three dimensions of error-handling practices during class discussion of mathematics tasks emerged. Teachers' ways of responding to student difficulties during such discussions varied in the extent to which they intentionally incorporated students' flawed solutions, addressed student errors in conceptually supportive ways, and mobilized students as a community of learners when errors were the focus. The relationship between each dimension of teacher response and teacher beliefs and knowledge is summarized

Table 4
Dimensions of Teachers' Error-Handling Practices During Class Discussion and Their Relationship to Teacher Beliefs and Knowledge

Dimension	Relationship to teacher beliefs	Relationship to teacher knowledge
Teachers differ in the extent to which they intentionally make flawed solutions a focus of whole-class discussion.	Some teachers avoided discussion of flawed solutions because they believed students would be embarrassed to have their mistakes shared publicly or that emphasis on errors would confuse students. Teachers who intentionally focused on flawed solutions believed that focusing on errors provided opportunities for learning that would benefit the individual and the class.	Engaging students in analysis of flawed solutions required teachers to draw on their knowledge base to emphasize mathematics concepts and confront misconceptions. Routines and scripts for organizing discussion of errors were also highly supportive of this practice.
Teachers differ in the extent to which their ways of responding to students' errors promote conceptual understanding.	Teachers who emphasized mathematics concepts were more likely to believe that understanding mathematics concepts is more powerful and more generative than remembering mathematics procedures.	Teachers with stronger knowledge of school mathematics were more likely to anticipate student errors and emphasize mathematics concepts as they responded to errors. Teachers with weaker knowledge struggled to unpack the underpinnings of errors. In responding to students, they were more likely to focus on procedures for obtaining correct answers without illuminating important mathematics concepts.
Teachers differ in the extent to which they mobilize students as a community of learners when errors emerge in class discussion.	Teachers who believed that their students were limited in their capacity to support each other's learning were less likely to encourage student collaboration. Additionally, teachers who viewed the purpose of class discussion as showing ways to get answers (a procedural focus) were less likely to press students to discuss the conceptual underpinnings of mathematical ideas.	Engaging students as a community of learners draws on knowledge of routines to promote productive debate. Further, this practice relies on a teacher's ability to interpret student contributions and steer the discourse in productive directions.

in Table 4. Each of these dimensions will be discussed in turn, with a focus on their relationship to teachers' beliefs and knowledge.

Dimension 1: Intentional Focus on Flawed Solutions in Whole-Class Discussion

The first way in which teachers' error-handling practices vary is the extent to which students' flawed solutions are intentionally selected for focus in whole-class discussion. Kazemi and Stipek (2001) identify the practice of having students analyze flawed solutions in whole-class discussion as a sociomathematical norm associated with classrooms in which students have high levels of mathematical understanding. When class discussion involves collaborative analysis and revision of flawed solutions, there are often opportunities for students to think about underlying mathematics concepts (Borasi, 1994; Leinhardt & Steele, 2005). Additionally, intellectual risk taking is promoted because mistakes are viewed as opportunities for learning. Of the 4 teachers studied, Ms. Rosena is the only teacher who intentionally integrated discussion of flawed solutions into whole-class discussion. The other 3 teachers aimed to orchestrate class discussion so that the focus was on multiple correct solutions, or they called on volunteers, with limited awareness of the strategies that would be shared.

Reflecting findings of other studies (Santagata, 2005; Silver et al., 2005), some teacher participants in this study expressed concerns about discussion of flawed solutions in the whole-class forum. This reluctance to discuss flawed solutions appears to be most closely related to teachers' beliefs. All teachers studied—except Ms. Rosena—expressed concern that exposure to flawed solutions might confuse children who have fragile mathematical understanding. Similar to the other teachers in this urban school, Ms. Rosena believed that her retained class of students had significant academic deficits. However, she also believed strongly that making the examination of errors a focus of class discussion was essential, because it helped students build robust mathematical knowledge from their own and their classmates' errors. She believed that if she did not address students' errors directly, they would continue to learn within a framework of limited understanding.

Teachers who did not intentionally incorporate errors into class discussion also worried that students might be embarrassed if their mistakes were "called out" publicly. Ms. Larsano intentionally avoided highlighting instances in which students were "wrong," because she did not want to negatively affect her ELL students' fragile self-confidence, self-esteem, and willingness to share ideas with the class. In contrast, Ms. Rosena's strong commitment to making flawed solutions a focus of whole-class discussion reflected her beliefs in the value and benefit of this practice. She convinced students that errors were a natural part of learning and that there was no shame in making errors, as long as one learned from them.

Although it appears that teachers' beliefs primarily influenced whether they intentionally chose to include analysis of flawed solutions in whole-class discussion, knowledge of how to lead productive discussion of student errors also is likely to have played a role. The professional development efforts in which the case study teachers participated focused on having teachers generate and unpack multiple correct ways of solving different kinds of problems, with little attention given to exploring flawed solutions. This professional development practice may have inadvertently contributed to teachers developing limited routines for responding to flawed solutions.

Dimension 2: Promotion of Conceptual Understanding Through Discussion of Errors

A second way in which teachers' error-handling practices in the context of class discussion vary is the extent to which they promote conceptual understanding of mathematics. Students who develop conceptual understanding of mathematical ideas can use their knowledge more flexibly than children who rely on procedural knowledge alone (Carpenter & Lehrer, 1999). Conceptual understanding of mathematics is supported when teachers emphasize mathematics concepts and productive problem-solving practices (Franke et al., 2007). In the instructional episode shared, Ms. Rosena stressed the concept of equal groups and continually referenced the problem context in conceptually supportive ways. In contrast, Ms. Larsano failed to provide an explanation that consistently reflected the problem context or illuminated key mathematical ideas inherent in the flawed solution.

The likelihood that teachers will respond to student errors during class discussion in ways that promote conceptual understanding seems highly related to the four facets of knowledge that undergirded my analysis: knowledge of relevant mathematical concepts, knowledge of student strategies and misconceptions, knowledge of teaching strategies to support development of conceptual understanding of particular mathematical concepts, and the ability to use mathematical knowledge to interpret student work in the moment. When case study teachers emphasized key mathematics concepts in response to student errors, they often included the mathematical ideas they intended to stress when describing their lesson image in preobservation interviews. Additionally, teachers who consistently emphasized mathematics concepts either developed premeditated scripts to respond to anticipated student errors or were able to successfully interpret student errors in the moment and formulate clear, conceptually based questions and explanations. In contrast, teachers with a weaker base of mathematical knowledge were less likely to identify key mathematical ideas as part of their lesson image, and they anticipated comparatively fewer student solution pathways and errors. Within lessons, these teachers sometimes struggled to understand students' errors and they regularly had difficulty formulating conceptually based questions and explanations in response to errors. As a result, teachers with weaker mathematical knowledge often addressed errors in conceptually unproductive ways, sometimes leaving errors unresolved. Often, these teachers reverted to greater focus on procedures, which tended to be more closely aligned with their personal ways of understanding mathematics.

Beliefs, on the other hand, seemed to play the greatest role in teachers' intentions to emphasize conceptual understanding. As the year progressed, Ms. Rosena and Ms. Jarmin both became more committed to reform-oriented beliefs that placed value on conceptual understanding. Evidence of their commitment was shown by the before and after scores on the IMAP beliefs survey relating to beliefs that pertained to conceptual understanding (Beliefs 2, 3, & 4). In preobservation interviews, both teachers consistently expressed a desire to encourage conceptual understanding. Whereas Ms. Rosena was usually able to utilize her knowledge for teaching mathematics to realize this intention, Ms. Jarmin often fell short due to difficulties interpreting the basis of students' errors and difficulties devising in-the-moment explanations and questions that were conceptually supportive. In contrast to these teachers, Ms. Larsano's before and after scores on the IMAP beliefs profile provided weak evidence of reform-oriented beliefs related to conceptual understanding. Although Ms. Larsano did express an intention to develop conceptual understanding of multiplication and division in relation to models and real-world situations in introductory lessons, her instructional goals quickly shifted to a focus on memorization of basic facts and algorithmic procedures for multidigit algorithms. For multidigit operations in particular, she said that being able to apply the steps of the standard procedures was more important than understanding why it worked. Ms. Larsano's weak adherence to reform-oriented beliefs about the role of conceptual understanding at times limited her efforts to emphasize conceptual understanding in her teaching.

Dimension 3: Mobilization of a Community of Learners to Address Errors

A third way in which teachers' error-handling practices in class discussion vary is in the extent to which they respond to errors by mobilizing students as a community of learners. Knowledge of routines and scripts to initiate and facilitate student collaboration is necessary to engage students as a learning community in a reform-oriented classroom. In this study, how much teachers utilized practices that engaged students in this way seems bounded by their knowledge of these routines, their mathematical knowledge, and their beliefs. When student difficulties surfaced in the context of whole-class discussion, only one of the 4 teachers—Ms. Rosena—utilized the reform-based practice of engaging students as members of a learning community to address difficulties. Especially in spring lessons, Ms. Rosena was observed prompting students to analyze and evaluate their peers' mathematical thinking, whether those students' mathematical ideas were correct or flawed. In the case of a flawed solution, Ms. Rosena guided the class to identify the source of the error and then to suggest ways to resolve the error. In contrast, when errors surfaced during class discussion in the other three classrooms, discourse typically followed traditional interaction patterns characterized by the teacher maintaining tight control of discourse and limiting the extent to which students worked together to analyze their peers' mathematical ideas, to judge mathematical correctness, and to resolve mathematical errors (Franke et al., 2007).

These different ways of responding to student difficulty fit well with the distinction Wood et al. (2006) make between discourse patterns in reform-oriented classrooms characterized by a *strategy-reporting* classroom culture and those characterized by an *inquiry/argument* classroom culture. Similar to the findings of

Wood and colleagues, the class discussion of student difficulties in Ms. Rosena's classroom, which come the closest to establishing an inquiry/argument classroom culture, seems to be the most productive of those observed. Yet, by her own account, Ms. Rosena believed this to be the most challenging aspect of reforming her own mathematics instruction and one on which she still needed to work.

In general, class discussions characterized by student inquiry and argumentation are more complex than discussions focused on strategy reporting. This greater complexity demands skillful application of deep and flexible teacher knowledge, including knowledge of mathematics, children's thinking, pedagogical routines for managing classroom discussion, and scripts for addressing particular mathematical content in productive ways. Furthermore, the ability to orchestrate a class discussion characterized by an inquiry/argumentation classroom culture (Wood et al., 2006) relies on a teacher's ability to interpret and assess student contributions in the moment to steer the discourse productively. In this study, the teachers appeared to find it more difficult to facilitate student inquiry into each others' strategies during discussions because they were expending their cognitive effort on simply making sense of and responding to student contributions. Even when Ms. Aria, Ms. Jarmin, Ms. Larsano, or Ms. Rosena fully understood students' strategies, they struggled to move away from the familiar routines of teacher-dominated discourse patterns. At other times, the teachers' efforts to mobilize discussion of mathematics among a community of learners were sidelined because the teachers' energy was consumed by classroom management issues.

One challenge facing urban schools that are shifting instruction to a reform-based mathematics curriculum is the higher degree of difficulty required for mobilizing a community of learners to address errors. Even for teachers such as Ms. Rosena, who recognized the benefits of this approach early in the year, making the necessary changes required a concentrated effort. This study suggests that a teacher's inclination to have students work as a community of learners to resolve mathematical errors that arise in class discussion can also be barrier, one that appears to be related to a teacher's beliefs about students' capabilities and the role of the teacher. Although Ms. Rosena's class of retained students and Ms. Larsano's class of ELL students were both composed of at-risk students, these teachers viewed their students' capabilities quite differently. Ms. Rosena expressed the belief that her students were capable of making positive contributions to the learning of others, and she viewed her role as one of a facilitator to help students make sense of each others' ideas. On the other hand, Ms. Larsano viewed her students as limited in their capability to solve problems or to contribute substantively to each other's learning during class discussion. She established herself as the dominant source of knowledge in the classroom, because she believed strongly that her students needed her to provide step-by-step instruction to learn to solve mathematical problems. This view of the roles of teachers and learners leaves little room for the substantial student contribution during class discussion of mathematics required by a reform-based program to realize the full benefits of this approach. However, it is important to note that Ms. Larsano also speculated that some other students—ones with less fragile understandings than her own—might be capable of playing a stronger role in class discussion, opening the door to the possibility that her views of the roles of teachers and students were highly situated in her urban school context and with her ELL students.

Another way that beliefs can influence the efficacy of a reform-based program is through the beliefs that teachers in an urban setting may have about the very purpose of class discussion, as occurred in this study. Thompson, Philipp, Thompson, and Boyd (1994) found that some teachers viewed class discussion as a venue primarily for children to learn ways to get answers. Whereas Ms. Rosena was using class discussion to develop the concept of equal grouping, Ms. Larsano was primarily interested in helping students develop successful procedures for solving division problems. If a teacher is not focused on using class discussion to focus on conceptual understanding, the potential content of class debate of mathematical ideas is limited.

In summary, study findings suggest that although beliefs and knowledge both influenced teachers' error-handling practices during class discussion of mathematics, certain aspects of instruction were more greatly influenced by teacher beliefs, whereas others were more greatly influenced by teacher knowledge. In brief, teacher beliefs appear highly related to the ways in which teachers structure class discussions when errors surface, including the roles they take on for themselves and ascribe to students. Teacher knowledge seems to be the primary determinant of the mathematical and pedagogical quality of teachers' responses to student errors during class discussions.

STUDY LIMITATIONS AND DIRECTIONS FOR FUTURE RESEARCH

Although the findings of this study provide insight into the interaction of teachers' knowledge, beliefs, and mathematics teaching practices, it is important to note that this study has several limitations. First, it is limited in scope. This study focuses on 4 teachers in a particular school setting primarily when the focus of instruction is on multiplication and division. Future research should consider more fully whether teachers' error-handling practices during class discussion and associated teacher knowledge and beliefs hold across settings and during instruction on other mathematics topics. Second, this study makes claims about how teachers' actions may have facilitated or limited student learning opportunities. However, evidence of student learning is limited to that which can be gleaned from observations and teacher interviews. In a research project with greater scope, it would be helpful to include a component more fully dedicated to determining what students are learning from particular instances of instruction. Third, the data collection and analysis for this research were carried out by one researcher. Although every effort was made to collect and analyze data in a consistent and unbiased manner, a research project of greater scope would be strengthened by coordinating the observations and interpretations of multiple researchers to allow for data to be examined from a wider range of perspectives and to increase the credibility of findings.

CONCLUSIONS

Current reforms in mathematics education advocate instruction that emphasizes classroom discourse that builds on students' thinking, promotes conceptual understanding, and mobilizes students as a community of learners. What we are beginning to discover, as schools strive to transition to reform-based mathematics, is that teachers' knowledge and beliefs can make a difference in how much change we can expect and how soon that change might occur. There is a real and immediate need to understand the role teachers' beliefs and knowledge play in how they engage students in class discussions. This is especially true in challenging urban school settings in which more students are struggling and in which there may be more barriers to success, including teacher perceptions related to instruction for their students.

By exploring how the beliefs and knowledge of teachers in one urban school may have influenced their responses to student errors during class discussion of mathematics, this study constitutes one step toward finding the appropriate professional development and support for teachers who have accepted the responsibility for preparing students to succeed in mathematics at schools that are shifting to a reform-based model. Findings from this study affirm the interactive relationship between teachers' beliefs and knowledge with regard to their influence on instructional practices. There is evidence suggesting that aspects of teachers' beliefs and knowledge contributed to each error-handling response pattern identified, with the relative influence of beliefs and knowledge varying based on the particular dimension of error-handling involved. In short, teacher beliefs seemed most related to the ways in which teachers structured class discussions when errors surfaced, whereas teacher knowledge appeared to drive the quality of teachers' responses to student errors in such discussions.

Study findings also suggest that teachers would benefit from greater awareness of common student errors and how these errors are related to key mathematics concepts. Furthermore, teachers particularly need support with envisioning how students' errors can be productively used as springboards for inquiry in the context of class discussion. To this end, professional development efforts in mathematics could be organized to deliberately model teaching practices for incorporating student errors in class discussion and make explicit the benefits of doing so.

This research complements other studies in highlighting the complexity of reform-based mathematics teaching that builds on student thinking, and it reinforces the assertion that it is difficult for teachers to make the kinds of changes envisioned by reformers. In addition to needing reform-based teaching materials, teachers need time, intellectual space, and human support to critically examine traditional mathematics teaching practices and assumptions about student learning in order to inspire recognition of the need for alternative mathematics teaching practices and to initiate changes in beliefs. To implement reform-based mathematics pedagogy well, many teachers require support that facilitates expanding and deepening their knowledge of school mathematics, knowledge of children's mathematical thinking, and knowledge of routines and scripts to enact such pedagogy. Although progress has been made in understanding the complexity of teachers' work engaging in reform-based mathematics teaching, there is much left to be learned about how to support and sustain teacher transitions to this pedagogical approach.

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