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Teaching and Learning Fraction Addition on Number Lines

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We present a case study of teaching and learning fraction addition on number lines in one 6th-grade classroom that used the Connected Mathematics Project *Bits and Pieces II* materials. Our main research questions were (1) What were the primary cognitive structures through which the teacher and students interpreted the lessons? and (2) Were the teacher's and her students' interpretations similar or different, and why? The data afforded particularly detailed analyses of cognitive structures used by the teacher and one student to interpret fractions and their representation on number lines. Our results demonstrate that subtle differences in methods for partitioning unit intervals did not seem important to the teacher but had significant consequences for this student's opportunities to learn. Our closing discussion addresses knowledge for teaching with drawn representations and methods for examining interactions between teachers' and students' interpretations of lessons in which they participate together.

Key words: Case study methods; Classroom interaction; Fractions; Middle grades, 5–8; Representations, modeling; Teacher knowledge

Research in education and psychology has demonstrated that adults and children often interpret joint activity in different ways, but little research has investigated the consequences of such differences for classroom teaching and learning. In separate subfields, researchers have gained significant insights into teachers and into students, but researchers have yet to analyze interactions between teachers' and students' interpretations of lessons in which they participate together. Without such analyses, education researchers have had to rely on less direct inference when investigating classroom teaching and learning and how it might be improved. This case study examines teaching and learning fraction addition on number lines in one sixth-grade classroom that used the Connected Mathematics Project (CMP) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002) materials. The main research questions were (1)

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What were the primary cognitive structures through which the teacher and students interpreted the lessons? and (2) Were the teacher's and students' interpretations similar or different, and why? Constructing methods for answering such questions was an important component of the study.

The data we collected afforded particularly close analyses of the teacher and one student. We present our results in three sections—solution patterns in the observed lessons, cognitive structures the teacher and student used to interpret fractions, and cognitive structures the teacher and student used to establish a partitioned linear unit. We will use these analyses to explain why the teacher and student interpreted key aspects of the observed solutions differently and will demonstrate that the instruction did not fully harness this student's nascent capabilities. We first review research that informed the study and, in so doing, identify theoretical constructs that we use.

BACKGROUND

The present study is informed by recent research that coordinates analyses of collective classroom mathematical practices and individual cognition, that connects teachers' knowledge and students' opportunities to learn, and that examines students' understandings of fractions.

Classroom Mathematical Practices and Individual Cognition

As part of their design research, Cobb and colleagues (e.g., Bowers, Cobb, & McClain, 1999; Cobb, 1999; Cobb, Stephan, McClain, & Gravemeijer, 2001; McClain & Cobb, 2001) have developed the emergent perspective that coordinates social and individual psychological analyses. Classroom communities are described in terms of social norms, sociomathematical norms, and classroom mathematical practices. Social norms include explaining and justifying in any domain, sociomathematical norms include what count as different mathematical solutions, and classroom mathematical practices are taken-as-shared ways of reasoning and arguing about particular mathematical ideas. A reflexive relationship between the learning of classroom communities and that of individuals is central to this perspective, but the published studies have focused more on emerging norms and practices than on the learning of individual students. Other researchers have also coordinated analyses of classroom mathematical practices and individual cognition using theoretical perspectives that emphasized conjecturing, experimenting, reasoning, and revising activities (Hall & Rubin, 1998) and focusing phenomena (Lobato, Ellis, & Muñoz, 2003). With the exception of Lobato et al., teachers in these studies either were education researchers or worked closely with researchers to design lessons at the center of the studies.

We use a modified definition of classroom mathematical practices better suited to studying the classroom in the present study, in which the teacher had no special relationship to the research community and did not elicit and build on students' ideas to the same extent as did teachers in many of the studies mentioned above. We define *classroom mathematical practices* as repeatedly observable solution patterns discussed by the teacher and students, a definition that places less emphasis on the taken-as-shared aspect used by Cobb and colleagues (e.g., Bowers et al., 1999; Cobb, 1999; Cobb et al., 2001; McClain & Cobb, 2001). The present study extends those mentioned above by analyzing cognitive structures through which the teacher and one student interpreted one practice as defined here.

Teachers' Knowledge and Students' Opportunities to Learn

Discussions of teacher knowledge have often been framed in terms of subject matter, pedagogical, and pedagogical content knowledge (e.g., Borko & Putnam, 1996; Shulman, 1986). Ball, Lubienski, and Mewborn (2001) argued that a critical next step for improving mathematics education is to examine not only the contents of teacher knowledge but also how teachers use their knowledge as they engage in their work. As part of this effort, a variety of complementary approaches have begun to shed new light on links between the knowledge of teachers and that of students.

Ball and colleagues (Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004) have been developing survey instruments for measuring knowledge that elementary and middle school teachers might use when puzzling about the mathematics in a student's idea, analyzing a textbook presentation, or considering the relative value of two different representations in the face of a particular mathematical issue. Hill et al. (2005) reported a positive correlation between first- and third-grade teachers' performance on the instruments and student gains on CTB/McGraw-Hill's Terra Nova measures of student achievement over a 1-year period in urban and suburban schools serving higher poverty populations. Linking teacher knowledge, as measured by the instruments, and student achievement was an important contribution, but this line of work leaves unaddressed questions about interactions between teachers' and students' knowledge, at the level of cognitive structures, during instruction.

Further research (e.g., Borko et al., 1992; Cooney, Shealy, & Arvold, 1998; Schoenfeld, 1998, 2000; Thompson, 1984, 1992) has used qualitative methods to examine how knowledge and beliefs together shape teachers' practices and, by implication, students' opportunities to learn. Some studies (e.g., Cooney et al., 1998; Thompson, 1984) have described broad relationships between teachers' practices and their beliefs about mathematics, teaching, and learning. Others (e.g., Schoenfeld, 1998, 2000) have provided utterance-by-utterance analyses of knowledge, goals, and beliefs working together to shape teachers' decisions during high school mathematics and physics lessons. Borko et al. (1992) examined Ms. Daniels's response when a student asked for an explanation of the invert-and-multiply rule for dividing fractions. Ms. Daniels's knowledge was insufficient to respond to the student in ways concordant with her professed beliefs about learning.

In the present study, we use the term *cognitive structures* as an umbrella that includes knowledge, beliefs, and goals. Like Borko et al. (1992) and Schoenfeld (1998, 2000), we examine how for one teacher a range of structures functioned together during instruction, and we add an analysis of cognitive structures used by one student to interpret the same lessons. The only studies of which we are aware that focus on teacher and student interpretations of the same lessons come from teaching experiments with one or two students (Thompson & Thompson, 1994; Tzur, 1999).

Students' Understandings of Fractions

Extensive research on students' reasoning with fractions has documented several understandings that also emerged as central to the present study, including interpreting fractions as pairs of whole numbers, maintaining (or not maintaining) a fixed whole, and operations related to partitioning. Several studies (e.g., Ball, 1993; Mack, 1990, 1993, 1995; Streefland, 1991) have reported elementary students' tendency to interpret numerators and denominators as pairs of whole numbers in which the denominator denotes the cardinality of the set and the numerator, the cardinality of a subset. This perspective can direct attention toward counting pieces in a partition of the unit and away from the multiplicative relationship between the size of a fraction and that of the unit. Researchers have also reported cases in which students struggled to maintain the correct referent unit when using this interpretation. For instance, students have misinterpreted two units, each divided into four pieces, as one unit divided into eight pieces (Ball, 1993, pp. 165–166; Mack, 1990, pp. 22–23).

Other researchers (Davis, Hunting, & Pearn, 1993; Hunting, Davis, & Pearn, 1996; Moss & Case, 1999; Olive, 1999; Olive & Steffe, 2001; Steffe, 2001, 2003, 2004) have examined how students' whole-number knowledge can support the construction of fraction knowledge. Olive and Steffe have conducted teaching experiments in which elementary students modified their knowledge of whole-number counting and multiplication to construct knowledge of fractions when solving tasks that involved lengths and areas. Differences in students' ability to coordinate units and the contexts in which they engaged their existing disembedding, iterating, and partitioning operations led to differences in the fraction schemes and operations they constructed. The most successful students constructed the operations of recursive partitioning and splitting.

We focus on iterating and recursive partitioning because they emerged as central when we analyzed our data. *Iterating* (Olive & Steffe, 2001; Steffe, 2001) involves taking a segment and joining copies end to end to recreate an already established unit. For instance, students might iterate to test an estimate for the length of 1/4 of a unit segment. *Recursive partitioning* (Steffe, 2003, 2004) is defined to be taking a partition of a partition in the service of a nonpartitioning goal. For instance, to understand the result of taking 1/3 of 1/4, students might begin by partitioning a unit into four pieces and continue by partitioning the first of those pieces into three further pieces. Determining the size of the resulting piece is a nonpartitioning goal, and students could accomplish this in several ways. Students might iterate the resulting piece and see that 12 copies reconstruct the original unit. This solution requires decomposing an initial unit into a unit of units (one unit containing 12 twelfths). Alternatively, students might recursively partition by subdividing each of the remaining fourths into three pieces. In contrast with the first solution, recursive partitioning involves decomposing an initial unit into a unit of units of units (one unit containing 4 fourths, each of which contains 3 twelfths). Other researchers have also reported that taking partitions of partitions has been central to students' construction of fraction concepts (e.g., Fosnot & Dolk, 2002; Kieren, 1990; Streefland, 1991, 1993).

Maintaining a fixed whole, interpreting fractions as pairs of whole numbers, iterating, and recursive partitioning are cognitive structures that played central roles in how the teacher and one student in the present study interpreted lessons on fraction addition. A main result was that the student's interpretation of fractions as pairs of whole numbers undermined her maintenance of a fixed whole. This led the student to interpret the teacher's demonstrated partitioning activity in ways that the teacher did not intend. Consequently, the instruction did not harness fully the student's nascent capabilities.

THE CoSTAR PROJECT AND MS. REESE

The present study was conducted as part of the Coordinating Students' and Teachers' Algebraic Reasoning (CoSTAR) project. The project conducts coordinated research on teaching and learning at Pierce Middle School, which in 2001 replaced traditional instructional materials focused on computational skill with the CMP materials. Pierce Middle School is in a rural community outside a large southern city, has approximately 700 students, and is racially and economically diverse. Prior to the present study, which took place in spring 2003, teachers at Pierce Middle School had limited professional development opportunities to support their transition to reform-oriented materials. The research began with fraction arithmetic in Grade 6 and has worked toward linear functions in Grade 8. The purpose of the project is not to evaluate the CMP materials nor to study implementation of reform but rather to study teachers' and students' interpretations of lessons in which they participate together.

This study examined teaching and learning in Ms. Reese's[1] 6th-grade classroom. Ms. Reese was confident in her teaching because of her background teaching high school algebra, praise she had received from colleagues and school administrators for her teaching, and a new job as a lead high school teacher that she would begin in fall 2003. At the time, Ms. Reese was in her 2nd year teaching 6th grade and teaching with the CMP materials. Prior to that, she had taught algebra for approximately 10 years, primarily to 7th-, 8th-, and 9th-grade students. Ms. Reese reported that her high school classes had focused on "traditional mathematics" and algorithms, had rarely used manipulatives, and had included drawn pictures only occasionally to introduce a topic or when there was confusion. The one place that

[1]All names are pseudonyms.

Ms. Reese did report using a more hands-on approach was in a class for 11th-grade students who had failed algebra in 9th and 10th grades. We approached Ms. Reese because her confidence and motivation suggested that she would be willing to discuss her practice with us, including moments during lessons when she and her students struggled. Moreover, she was interested in learning more about how her students interpreted her instruction. These characteristics were essential for gathering data that would help address our two main research questions.

Although skeptical at first whether CMP would "work," Ms. Reese reported that she was now "sold" on the materials. From our perspective, she seemed earnest about using the CMP materials successfully. She followed the sequence of lessons, making adjustments that she felt would improve her students' opportunities to learn. She focused on mathematics for the entire 50-minute period and reported incorporating group work into her teaching for the first time. Ms. Reese explained, however, that when planning, she did not "sit there and read every word" of the teacher's edition but rather scanned to get an idea of what she and her students were supposed to do. After that, she "put the teacher's edition up" and taught. From our perspective, Ms. Reese's practice combined aspects of more traditional with aspects of more reform-oriented instruction.

BITS AND PIECES II, RECURSIVE PARTITIONING, AND COMMON DENOMINATORS

During the present study, Ms. Reese was implementing the *Bits and Pieces II* unit for the first time, using a draft revision of the published unit (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2003). The preceding unit, *Bits and Pieces I*, develops fraction-strip, number-line, and area representations of fractions; connections among fractions, decimals, and percents; equivalency; and order. Fraction strips are rectangular strips of paper subdivided by either folding or drawing marks. *Bits and Pieces II* develops fraction arithmetic through problems in which fractions are embedded. Teachers are to help students develop their own strategies and to inject further ideas for students to consider. Overarching goals for the unit include learning when to use each operation and how to estimate answers using 0, 1/2, and 1 as *benchmarks* or reference points. With respect to fraction addition and subtraction, the focus of the present study, the introduction to the teacher's edition states that (a) students learn to combine quantities in contexts and write mathematical sentences using symbols to indicate required computations, and (b) students' previous experiences partitioning line segments representing one whole are critical as they learn to change representations of fractions to forms with common denominators.[2]

The draft *Bits and Pieces II* revision does not spell out how students could use previous experiences partitioning line segments representing one whole to construct common denominators. We fill in some details. The fraction-strip activities in *Bits*

[2]The published second edition (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006) dropped line segments and the number line from the fraction addition lessons.

and Pieces I afford opportunities to develop fractions as linear quantities and partitioning as an operation that divides unit lengths into equal-sized pieces. Students could extend these experiences by taking partitions of partitions to generate equivalent fractions—for instance, by partitioning halves into three equal-sized pieces and seeing that 1/2 is the same length as 3/6. Finally, for students to develop computational procedures based on common denominators, they would need to coordinate up to three partitions. For example, to measure a length that combined 1/2 and 1/3, students would need to find a new partition of the unit that simultaneously subdivided halves and thirds. Common partitions could then lead to equivalent fractions with common denominators.

METHODS

An important component of the CoSTAR project is the construction of methods for inferring cognitive structures through which teachers and students interpret lessons in which they participate together. We videotaped Ms. Reese's instruction every day during the same class period using two cameras. The present study focuses on the first 4 of 8 weeks that we taped. One researcher set the first camera in the back of the classroom and recorded the entire class, adjusting the levels of ceiling and wireless microphones to hear all students during whole-class discussion and to zero in on conversations between Ms. Reese and individual students during group work. A second researcher used the second camera to record written work, staying at the back of the classroom to record the whiteboard during whole-class discussion and shadowing Ms. Reese to record work that she discussed with students at their desks. Later the same day, we combined the video and audio from the two cameras using an audiovisual mixer to create a *restored* view (Hall, 2000) that captured much of what Ms. Reese and her students said, wrote, and gestured toward during the lesson. The choice to dedicate the second camera to written work reflects our emphasis on the role that inscriptions play in problem solving. We used the restored view to identify instances in which Ms. Reese and her students used fraction strips and number lines to understand fraction addition and subtraction. From these instances, we selected examples to use in interviews.

To gain more detailed access to the cognitive structures through which students interpreted Ms. Reese's instruction, the first author interviewed three pairs of students from the videotaped class period. Ms. Reese identified students that, in her opinion, reflected a cross section of achievement. One pair consisted of two high-achieving boys, one pair consisted of two low-achieving girls, and one pair consisted of a mid- to high-achieving girl and a low-achieving boy. The present report focuses on the pair of girls, Sonya and Jenny. Each pair was interviewed once a week for approximately 50 minutes starting the 1st week of the study. During the semi-structured interviews (Bernard, 1994, chapter 10), students worked problem-solving tasks using pencil and paper and watched lesson excerpts on a laptop computer. The tasks were similar to those at the center of the excerpts. Questioning students about their reasoning when working problems provided further access to the cognitive structures they used. Questioning students about the lesson excerpts provided access to their interpretations of related instruction. We made every effort to interview all three pairs of students within 4 days of the lesson(s) from which we took excerpts. Holidays and unexpected interruptions in the school schedule created occasional delays. The interviewer used two video cameras—one to record students and one to record written work and the computer screen—from which we created a restored view that captured much of what the students said, wrote, and watched. We selected excerpts for use in teacher interviews that revealed aspects of students' mathematical thinking and difficulties not evidenced during lessons.

Finally, a project member not listed as an author conducted weekly, hour-long interviews with Ms. Reese to gain access to her interpretations of the *Bits and Pieces II* activities, her understandings of her students, and the pedagogical decisions she made. The interviewer used a combination of lesson and student interview video excerpts selected in consultation with the first author and played on a laptop computer. The consultations ensured that many of the same lesson excerpts were used in student and teacher interviews. The teacher interviews occurred after we completed a cycle of student interviews, usually within a week of the original lessons. When playing the lesson excerpts, the interviewer asked Ms. Reese to discuss the mathematical content and to reconstruct her thinking as she used representations to support student learning. When playing the student interview excerpts, the interviewer asked Ms. Reese to respond to her students' problem solving and interpretations of lesson excerpts. These excerpts provided Ms. Reese more detailed access to her students' mathematical thinking than she could often gain when teaching a whole class. Ms. Reese's reactions to students' mathematical thinking gave us further access to her understanding of her students and the mathematics in *Bits and Pieces II*. The interviewer used two video cameras—one to record Ms. Reese and one to record the computer screen—from which we created a restored view that captured much of what Ms. Reese and the interviewer said and watched.

After data collection was complete, we conducted more detailed analyses using methods similar to those described by Cobb and Whitenack (1996); Izsák (2003); and Schoenfeld, Smith, and Arcavi (1993) for analyzing longitudinal sets of video recordings. We analyzed verbal references, hand gestures, and added inscriptions (e.g., shading and circling) for evidence of the representational features to which Ms. Reese and her students attended and how they used those features to accomplish steps during whole-class and group work. Our purpose was to identify observable behaviors that Ms. Reese and her students tried to make sense of, and we found a stable pattern in demonstrated solutions that will be presented in the results section below.

We then analyzed interview data in conjunction with lesson data for evidence of cognitive structures through which Ms. Reese and her students might have interpreted the lessons. First we went through the student interviews line by line to generate and refine descriptions of cognitive structures that students evidenced through talk, gesture, or inscription. We viewed and reviewed the interview videos until our descriptions became stable and could account for students' generation and use of number lines and fraction strips to solve fraction addition problems. This phase of analysis revealed the central role played in the present case by students' knowledge of whole numbers and partitioning. Although the interviews provided much more dense and detailed data on students' cognitive structures than did the lessons, we examined the lesson videos for further evidence against which we checked our descriptions. The present report concentrates on those structures most central for comparing how Ms. Reese and Sonya interpreted the lessons.

Ms. Reese's explanations during her *Bits and Pieces II* lessons provided substantial evidence for the understandings of fractions she used. We analyzed the teacher interviews for further evidence of her understandings of students and of content that shaped those explanations. In particular, we examined all instances in which Ms. Reese talked about the role of drawn representations in teaching and learning to add and subtract fractions. We created initial descriptions of Ms. Reese's goals and understandings and then examined the lesson videos to determine if her instruction evidenced similar goals and understandings. This increased our confidence that knowledge Ms. Reese evidenced during the interviews did in fact shape her instruction. We refined our descriptions of Ms. Reese's goals and understandings when we found instances from lessons or interviews that were inconsistent with our initial descriptions. This was particularly important given the delay of up to a week between the initial lesson taping and the following teacher interview. We went back and forth between the interview and lesson videos until our descriptions became stable and appeared to account both for Ms. Reese's use of fraction strips and number lines during lessons and for explanations of her practice that she provided during interviews.

When presenting the transcript, we label excerpts with letters. Alphabetical order corresponds to the presented order, not the chronological order, of excerpts. In preparing the transcripts, we used the following conventions:

(...) denotes a comment inserted while preparing transcripts.
[ ] denotes a word substitution.
//...// denotes concurrent talk.
italics denotes emphasis added.

RESULTS

Part I: Classroom Solutions to Fraction Addition Problems on Number Lines

The 19 students in Ms. Reese's class reflected the racial composition of Pierce Middle School and represented a broad range of achievement in mathematics and reading. At the beginning of the study, Ms. Reese reported that her students had a good grasp of fraction strips and equivalent fractions from the preceding *Bits and Pieces I* unit and were ready to develop numerical methods for fraction arithmetic. Moreover, some students had studied fraction addition in fifth grade, but most were making the common mistake of adding numerators and denominators (e.g., 1/2 +

1/3 = 2/5). She also reported explaining the CMP tasks to students because they had trouble reading the materials. On the whole, Ms. Reese perceived her students as motivated, even if some were "challenged" and others' attention drifted. Our perceptions of students' motivation were largely consistent with Ms. Reese's.

During typical whole-class discussions, Ms. Reese stood at the board and called on students as the class worked through problems. She often identified the next problem-solving step and asked students how to accomplish it. After a student made a suggestion, Ms. Reese often polled the class, asking various students if they agreed or disagreed. This routine gave students opportunities to provide reasons for agreeing or disagreeing and allowed Ms. Reese to control the direction of the solution. Ms. Reese switched to partner work when she felt that students needed to work with each other. After working with students at their desks, she pulled the class together for a wrap-up discussion before the period ended. Ms. Reese stated that she wanted students to tell her what to do next in a solution, but, from our point of view, she did not elicit and build upon students' ideas as fully as the CMP materials suggested.

Ms. Reese's main instruction on using number lines to add and subtract fractions occurred on March 24 and 25. Previous lessons had begun to develop fraction addition and subtraction using area and recipe contexts. The *Bits and Pieces II* materials now moved toward algorithms by having teachers and students first represent already completed problems on the number line. The text emphasized that fractions can be used to show locations on the number line or lengths measured from 0. During the lessons, Ms. Reese demonstrated how to represent four completed problems: 4 + 6 = 10, 1/4 + 1/8 = 3/8, 2/3 + 5/9 = 11/9, and 5/10 – 2/5 = 1/10.[3] When solving 4 + 6 = 10, she emphasized that number lines go on forever in both directions, that negative numbers are to the left of 0, and that the fraction strips with which the class had been working corresponded to the interval between 0 and 1.

Starting with the second example, Ms. Reese carried out a sequence of three steps. The pattern suggested that she wanted students to learn a particular method and gave students multiple opportunities to make sense of her demonstrated approach. The first step was to determine how many whole numbers to represent. In this example, 3/8 was between 0 and 1 (Figure 1a). In other examples, Ms. Reese had students estimate the sum or identify the minuend and round up to the next whole number. The second step was to subdivide the unit interval(s) created in the first step. Ms. Reese traced the interval from 0 to 1 with her finger and asked how to "divide up this amount." One student suggested a half; another suggested eighths. Ms. Reese took the second suggestion, saying that she needed "eight pieces that look about the same." She made seven tick marks from left to right (Figure 1b), labeling them 1/8, 2/8, . . . , 7/8. The third step was to draw arrows for each addend and circle the sum. Ms. Reese told the class that she could not see 1/4 on the number line, but a student pointed out that 2/8 was the same. Ms. Reese agreed and drew one arrow

[3]The first two problems in *Bits and Pieces II* were 3 + 2 = 5 and 3/4 + 1/8 = 7/8. We did not ask Ms. Reese why she used different problems, but the changes did not appear significant to us.

for each addend as she spoke: "OK. So start at 0 and go over to 2/8 and then go over 1/8 more." Ms. Reese circled the answer and concluded that 3/8 "is where you are supposed to land" (Figure 1c).

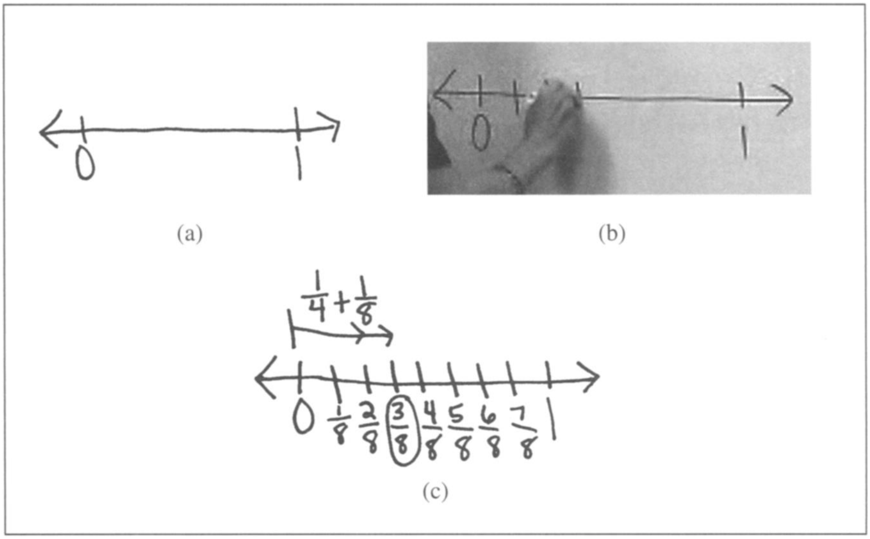

Figure 1. An example of the pattern in demonstrated solutions to fraction addition problems on number lines. (a) Determine how much of the number line to draw. (b) Subdivide the unit interval(s). (c) Draw arrows for each addend, and circle the answer. Original work in (a) and (c) has been retraced.

In subsequent examples, Ms. Reese had students compute equivalent fractions before finding them on number lines and consistently subdivided unit intervals by adding tick marks from left to right. She was skilled at estimating the spacing so that the final tick mark corresponded to the whole. In cases in which her final subinterval was slightly off, Ms. Reese moved the location of the whole to create equal-length subintervals.[4] During the solution to 2/5 + 1/10 = 5/10, Ms. Reese pointed out that she was moving the 1 and told students,

A1. Reese: I started putting pieces in here just eyeballing it and trying to space them out, and I didn't have enough spaces, so I moved the 1 over and made another mark because I need 10 spaces. . . . This is a space (pointing with thumb and index finger), and you need 10 of those to be called 10th-size pieces.

[4] Recall that Olive and Steffe (2001) and Steffe (2001) have labeled such actions as iteration but only when the unit has been established and remains fixed.

Finally, she consistently articulated the counting and landing perspective when drawing the second arrow and circling the sum. For our research questions, analyzing cognitive structures through which Ms. Reese and her students interpreted the three-step solution pattern was more important than gauging the extent to which her implementation did or did not align with the intentions of the curriculum developers. To foreshadow one result, although Ms. Reese may have focused on a fixed unit length even as she adjusted the location of the whole, this modification appeared to undermine unintentionally Sonya's attention to a fixed referent unit.

Part II: Ms. Reese's and Sonya's Interpretations of Fractions

This section summarizes the interpretations of fractions that Ms. Reese and Sonya appeared to use and presents evidence that Sonya's focus on pairs of whole numbers contributed to her misinterpretation of Ms. Reese's explanations involving fraction strips and number lines.

Ms. Reese Interprets Fractions as "Amounts"

An example from Ms. Reese's third lesson on estimating fraction sums provides evidence for understandings of fractions that she apparently used while teaching. The lesson occurred on March 12 and preceded her lessons on fraction addition described above. The problem asked students to check whether 7/12 + 5/8 = 23/24 was correct using estimation. Ms. Reese worked with students at their desks and noticed that many were calculating 12/20 incorrectly by adding numerators together and adding denominators together. She addressed the difficulty in a whole-class discussion. Ms. Reese referred to, but did not draw, fraction strips when reminding students that numerators tell the number and denominators tell the size of the pieces. She concluded her explanation by saying:

B1. Reese: Y'all saying top numbers and bottom numbers. Y'all, *these are amounts* (pointed to 7/12) and *this is 12th-size pieces and we got 7 of them.* That's the size, 12ths is the size of the piece and there are 7 of them. Y'all remember me telling you this when we did strips. We talked about it and talked about it. *This is 8ths-size pieces* (pointed to 5/8). These pieces are bigger than those, right? And *you've got 5 of them*, which means you have, if you put that together, you have a little more than a whole.

Ms. Reese used similar language several other times during the same episode and further lessons (recall her discussion of "10th-size pieces," line A1). She apparently referred to a fixed unit when she compared the sizes of 8ths and 12ths (line B1) but left this implicit. In interviews, Ms. Reese emphasized that the unit had to be fixed to compare fractions, that partitions consisted of equal-sized subintervals, and that students should focus on the spaces between tick marks. During one interview, Ms. Reese reviewed a video excerpt of her solution to 1/4 + 1/8 = 3/8 described above and talked about 3/8 as a length. From these data, we inferred that during her lessons Ms. Reese often focused on fractions as lengths described in terms of a number of equal parts in a fixed unit interval.

Sonya Interprets Fractions as Pairs of Whole Numbers

Ms. Reese identified Sonya and her interview partner, Jenny, as students who were often confused. She reported that Sonya often seemed to understand something at the end of one lesson, only to forget it the next, and that she had trouble explaining her thoughts. Data from the lesson and interview videos confirmed that both students did indeed experience significant difficulties with fractions and gave explanations that were difficult to follow. That the students' whole-number arithmetic was error prone contributed to their difficulties. Both students also evidenced the common mistake of adding fractions by adding numerators together and then adding denominators together, the same error to which Ms. Reese responded on March 12 (line B1). Sonya and Jenny could compute equivalent fractions correctly but had trouble generating equivalent fractions with common denominators: They typically multiplied the numerator and denominator of each fraction by 2 but did not choose readily different multipliers for each fraction. Thus, Sonya and Jenny had shaky knowledge of computational procedures that they accessed ineffectively. Our analysis will emphasize Sonya's structures because Jenny tended to listen first and then agree, making it hard to determine to what extent her answers simply imitated Sonya's.

Sonya's and Jenny's greatest source of difficulty when interpreting Ms. Reese's number-line solutions appeared not to be their computational procedures but their conception of fractions in which the denominator stood for the cardinality of the set and the numerator stood for the cardinality of a subset. We refer to this conception of fractions as *n out of m*. Although the *n*-out-of-*m* structure might be sufficient to represent some situations (3/4 can represent three out of four marbles), this structure made it hard for the students to maintain a fixed unit when reasoning with fraction strips or number lines.

The *n*-out-of-*m* structure surfaced during Sonya's and Jenny's first interview, which focused on estimating sums. The interview occurred March 17 after Ms. Reese had conducted four lessons on estimating sums but before she introduced fraction addition on number lines. As Sonya estimated 1/8 + 1 4/10, she drew two fraction strips to show 1 4/10 (Figure 2a) and explained:

C1. Sonya: This is like one whole of a piece (referred to the top strip), and then this is a 10th strip with 4 of them shaded in (referred to the bottom strip). Like 4 pieces, like it was 10 pieces of candy and 4 of 'em got eaten.

That Sonya shaded 4 disjoint regions in the second strip provided further evidence that she thought of 4 out of 10 pieces. Her *n*-out-of-*m* structure included thinking of the entire strip as the whole and of a single element within the strip as a second unit. Jenny also made comments consistent with the *n*-out-of-*m* structure.

A few moments later, the interviewer drew the outline of a fraction strip and had the students complete a halves strip (top strip in Figure 2b). He then asked for a thirds strip that was the same size. Sonya wanted to extend the existing halves strip with a third approximately equal-sized piece. Thus, she did not maintain the same length strip for halves and thirds. Jenny disagreed with Sonya and drew a new strip the

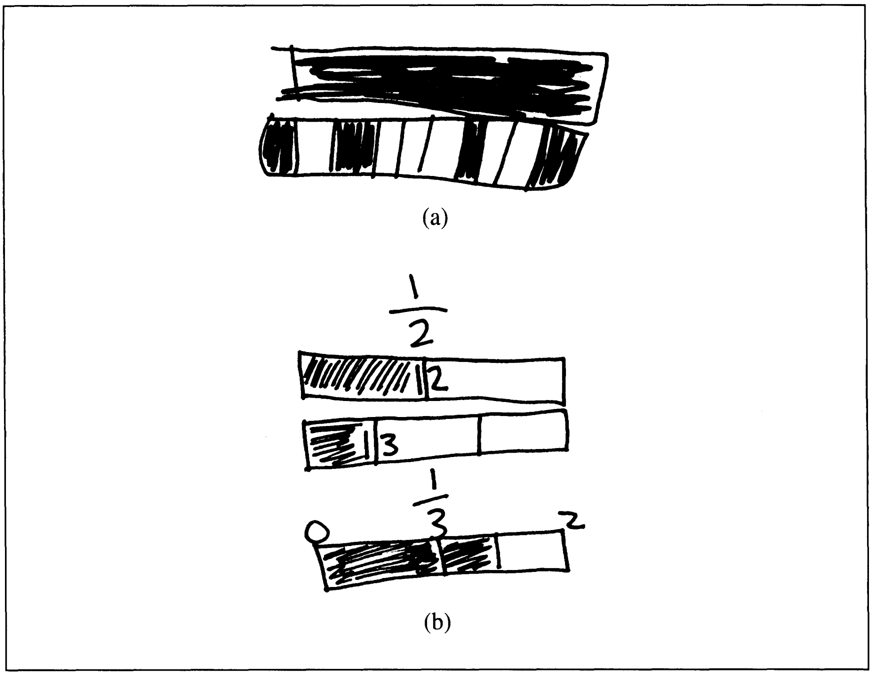

Figure 2. (a) Sonya's drawing of 1 4/10. (b) Sonya and Jenny's fraction strips for 1/2, 1/3, and 1/2 + 1/3. Original work in (a) and (b) has been retraced.

same size as the first (middle strip in Figure 2b). The interviewer asked the students to draw a third fraction strip the same size as the first two and to shade an estimate for 1/2 + 1/3. Jenny drew the outline. This time Sonya shaded half the strip and used her fingers to translate the shaded region on the original thirds strip to the new strip (bottom strip in Figure 2b). These data foreshadow those we present in a subsequent section which demonstrate that Sonya could attend to a fixed whole with support.

Sonya and Jenny estimated the sum of 1/2 and 1/3 to be between 1 and 2 and gave several explanations in which they referred to the midpoint of the bottom strip (Figure 2b) as "1" and the right-hand end as "2." When the interviewer asked what number corresponded to a whole strip, Sonya and Jenny gave a series of examples, including "6 over 6," "5 over 5," and "2 over 2" but not the number 1. In fact, the students pointed again to the middle of the bottom strip when identifying where they saw the 1. Apparently, Sonya and Jenny thought of whole strips as showing *n* unit elements from a set of *n* elements (further evidence for the *n*-out-of-*m* structure) and did not fully coordinate whole strips, the numeral 1, and expressions like "2 over 2." Instead, they connected the 1 with the first piece of their partitioned strip. Near the end of their second interview, Sonya evidenced similar interpretations of number lines when comparing 3/4 and 6/8. That she reasoned in similar ways across 2 different days, tasks, and linear representations suggested the robustness of her

n-out-of-m structure and made it unlikely that she would attend to a fixed unit, as implied by Ms. Reese's explanations (e.g., line B1).

Interpreting Ms. Reese's Instruction through N out of M

During their fourth interview, April 9, the interviewer asked Sonya and Jenny to solve 3 1/6 – 1 2/3 on the number line. The students remembered that Ms. Reese had discussed the problem in class (on March 25) and rewrote the problem correctly as 3 1/6 – 1 4/6. Sonya drew a number line and said, "You would want to have 6 on here because that's what the denominator is." She wrote "0" at the left-hand end and "6" at the right-hand end of her number line as she spoke. Sonya told Jenny to "draw the half line first," labeled Jenny's tick mark "3/6," and wrote "1" on the right-hand end under the "6." The students labeled the tick marks as shown in Figure 3a, and Sonya explained, "Six is the whole because 6 over 6." The students calculated the incorrect answer of 2 3/6 (subtracting 1/6 from 4/6 and 1 from 3). Sonya then added three tick marks to each interval (Figure 3b) and verbally assigned numbers to tick marks from left to right as follows: 1 1/6, 1 1/7, 1 1/8, 1 1/9, 2/6, 2 2/7, 2 2/8, 2 2/9, 3/6, 3 3/7, 3 3/8, 3 3/9, 4, 4 4/7, 4 4/8, 4 4/9, 5, 5 5/7, 5 5/8, 5 5/9, and the whole. In so doing, she assigned either whole numbers or numbers of the form $n/6$ to the original tick marks in Figure 3a.

The interviewer showed a still image of the number line that Ms. Reese drew when solving the same problem (Figure 3c). The students said that their number line was the same, just numbered differently, and continued:

D1. Jenny: We showed what the sizes were. We showed what the sizes of those wholes were.

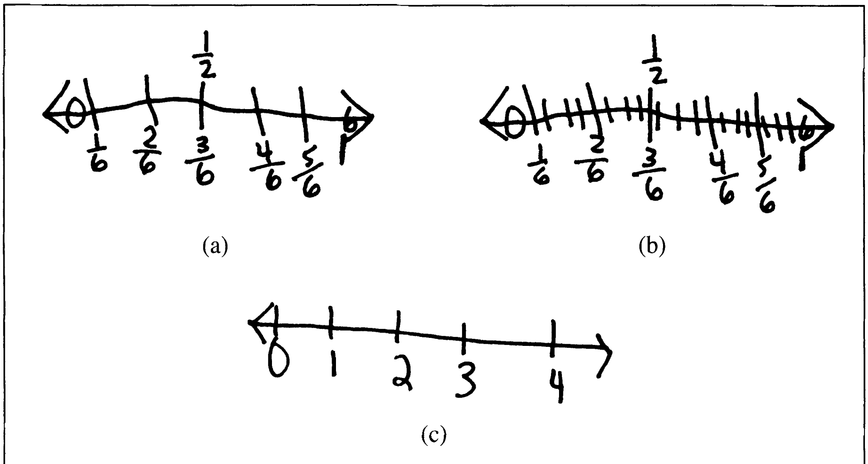

Figure 3. (a) Sonya and Jenny's number line. (b) Sonya adds tick marks. (c) Ms. Reese's number line. Original work in (a), (b), and (c) has been retraced.

D2. Sonya:	Yeah.
D3. Int:	You showed what the sizes of the wholes were.
D4. Jenny:	Yeah. Like that's *a whole in a sixth size* (pointed to 1/6), and that's a *2 wholes in a sixth size* (pointed to 2/6), and that's *//3 wholes in a sixth size* (pointed to 3/6).
D5. Sonya:	Right.// This is a *sixth-size strip*. And this is a 1, this is *one piece of a sixth-size strip* (pointed to 1/6).
D6. Int:	Mm.
D7. Sonya:	This is *two pieces of a sixth-size strip* (pointed to 2/6). This is a half of a sixth-size strip (pointed to 1/2). This is a fourth of a sixth-size strip (pointed to 4/6), and this is a fifth of it (pointed to 5/6), and this is a whole [strip] (pointed to 1).[5]

These data provided initial evidence that Sonya and Jenny assimilated Ms. Reese's explanations into their *n*-out-of-*m* structure. The language used by the students (lines D4-D7) closely resembled that used by Ms. Reese in line B1 ("Twelfths is the size of the piece, and there are seven of them"). As discussed above, Ms. Reese apparently referred to a fixed unit length but left this implicit. Sonya and Jenny also talked about denominators as a size but connected size with the number of wholes: A sixth-size strip was a strip with six wholes. This explanation was consistent with others we presented above as evidence for Sonya's *n*-out-of-*m* structure.

Further evidence came when the interviewer played the clip in which Ms. Reese subdivided each of the four intervals (Figure 3c) into sixths. When asked, "Where do you see one whole in [Ms. Reese's] picture?" Sonya pointed to the "4" to indicate the whole and to the "0" to indicate where the whole began. After the interviewer played more of the clip that showed Ms. Reese labeling her tick marks between 1 and 2, Sonya assigned numbers to the tick marks between 0 and 1 correctly and said that "6 over 6" was the same as the 1 on the number line. The interviewer pointed out that a few exchanges earlier, they had pointed to the "4," and Sonya commented that was "where the 4 over 4 was at." She then explained, "It's two wholes on here. Like it's two [labels] on here. Like zero, one, two, three, four (pointed to 0, 1, 2, 3, 4). These are already whole numbers." A few exchanges later, Sonya added, "It's two different sets of numbers here. It's whole numbers and then it's little lines in between the one whole numbers." Finally, she said. "We didn't know you mean where the 6 over 6 was at." Thus, Sonya appeared to apply her *n*-out-of-*m* structure twice, first to the interval from 0 to 4 and then to the interval from 0 to 1. Her *n*-out-of-*m* structure and incomplete connections among the whole, the numeral 1, and fractions of the form *n*/*n* apparently afforded her "two different sets of numbers" perspective. We did not show this excerpt to Ms. Reese because we gave priority to other episodes from the lessons and student interviews. A subsequent section of this article will demonstrate that Sonya could represent fractions appropriately once she focused on a fixed unit.

[5] Note that after the half, Sonya's language shifted from whole-number to fraction words. Although Sonya may have meant just the fourth piece was one fourth and just the fifth piece was one fifth, our data did not include other instances of Sonya reasoning like this explicitly.

Part III: Ms. Reese's and Sonya's Interpretations of Partitioning

In this section we analyze cognitive structures that Ms. Reese and Sonya used to establish partitioned unit intervals, the second of the three steps in the demonstrated pattern of solutions summarized above. We present evidence that Ms. Reese used recursive partitioning in special circumstances but that a coherent set of goals and understandings directed her attention more often toward partitioning from left to right and adjusting the location of the 1 if necessary. We then present evidence that Sonya assimilated Ms. Reese's demonstrations through structures that undermined her attention to a fixed unit. Finally, we demonstrate that Sonya began to reason about partitions of partitions appropriately once she focused on a fixed unit and, thus, that the instruction did not fully harness her nascent capabilities.

Ms. Reese's Attention to Recursive Partitioning

The first evidence that Ms. Reese attended to recursive partitioning came at the end of her third interview on March 26, approximately 2 weeks into her implementation of *Bits and Pieces II* and just after her fraction addition lessons summarized above. The interviewer had been probing Ms. Reese's strategies for subdividing unit intervals and had pointed out that "eyeballing" a half was easier than an eighth. When asked whether she discussed such issues with her students, Ms. Reese offered the following example from *Bits and Pieces I*: "When we did our fraction strips, and we did the folding, and we did the half, and then we folded it some more and made fourths, and then folded it some more and made eighths." Ms. Reese also said, "It wasn't something that we just went on and on about." We inferred from these comments that Ms. Reese had attended sometimes to creating partitions of partitions but had not emphasized this approach during her implementation of *Bits and Pieces I*.

A second example of Ms. Reese's attention to recursive partitioning occurred in class on April 7 as the class reviewed for a test. She asked students to estimate whether 1/5 + 1/10 was closer to 0, 1/2, or 1 and said that this task was hard. She drew a 5ths strip and shaded the first 5th. She then drew a 10ths strip directly underneath and emphasized that the two strips were the same size. She subdivided the 10ths strip from left to right, explained that two 10ths "fit" into each 5th, and shaded the first 10th. Finally, she located 1/2 on each strip and drew an arrow to indicate moving the shaded 10th next to the shaded 5th (Figure 4). When students' estimates for the sum included 0, 1/2, and 1, Ms. Reese said, "I am hearing so many different answers that I think what I'm going to have to do is split up my 5ths into 10ths." She then subdivided each 5th in the top strip shown in Figure 4, and students chorused that the sum was closer to 1/2. We interpreted Ms. Reese's demonstration as a form of recursive partitioning because she partitioned a partition in the service of estimating a sum.

A third example occurred in class on April 8 as Ms. Reese watched Sam use a number line to solve 1/3 + 2/9. Sam was one of the highest achieving students in the class. He drew a unit interval, made and labeled tick marks for 1/3 and 2/3, and then

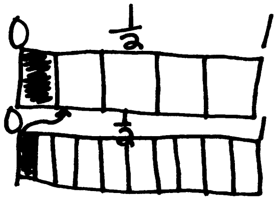

Figure 4. Ms. Reese's 5ths and 10ths strips. Original work has been retraced.

added tick marks for the ninths. Ms. Reese commented, "I thought that was interesting, [Sam], how you started out with thirds and then you went back and made ninths. OK. It works for me." Ms. Reese continued to watch as Sam determined the correct sum, 5/9. In the wrap-up discussion, Ms. Reese told the class that Sam had first drawn thirds and then divided his thirds into ninths, but she did not demonstrate Sam's method on the board. Rather, she added tick marks from left to right as she had done in her previous solutions and commented to the class that she had to adjust the location of her 1. When we played the lesson video clip of Sam's method for Ms. Reese in an interview on April 21, she said that Sam was "just real meticulous about his work looking good" so "he wanted to go ahead and make it one third and two thirds and then break those up into pieces so it would be more accurate."

The three examples just described provided reasonably good evidence for our claim that Ms. Reese attended to recursive partitioning (though she would not call it that) at some points during her instruction. Moreover, she could discuss the operation during interviews. Yet the lesson and interview data indicated that her instruction did not emphasize taking partitions of partitions. In addition to the examples on March 24 and 25, we found 11 more instances in which Ms. Reese drew fraction strips or number lines during lessons. These included the examples of 1/5 + 1/10 (Figure 4) and Sam's work discussed above. The example of 1/5 + 1/10 was the closest that Ms. Reese came to discussing recursive partitioning explicitly during the lessons we observed. Although Ms. Reese attended to recursive partitioning in special situations, such as ones calling for accurate pictures, a further coherent set of goals and understandings directed her attention more often toward partitioning from left to right. We describe these next.

Ms. Reese's Overarching Goals

Ms. Reese's implementation of the *Bits and Pieces II* materials appeared driven by three overarching goals. She stated two during her first interview: that students understand the sizes of fractions ("amounts") and that students come up with algorithms. Ms. Reese also discussed a third goal: avoid confusion.

The first evidence we present that Ms. Reese tried to avoid confusion came during the interview on April 8, just hours before she watched Sam solve 1/3 + 2/9. The interviewer asked Ms. Reese whether she ever used big hash marks for whole units and smaller hash marks for halves and further subdivisions. Ms. Reese said "No" and gave her reason:

E1. Reese: I think that's a more advanced method of the number line, that maybe we're not there in sixth grade. I don't know. That's just something I, *I choose to try to make it as simple as possible*. And just from teaching sixth grade last year, my first time, I found out that a lot of things that I explained and talked to them about was not necessary because *it caused a lot more confusion than understanding*. So *I try to just simplify it as much as possible,* and if they come up with the different size tick marks, great. But, I just, I think it would be good, but I don't want to . . . *I feel like it may be a little bit confusing for sixth grade.*

Further evidence that Ms. Reese sought simple methods for partitioning intervals came from an interview on April 21. The interviewer asked whether discussing Sam's method for solving 1/3 + 2/9 described above would have been useful for the entire class. Ms. Reese answered:

F1. Reese: That could be emphasized more but, like I said, it's hard to know how deep to go into which things to avoid . . . and I think my thoughts is . . . *I'm so afraid that I'm gonna confuse them deeper by, by offering too much too deeply.*

Ms. Reese continued by explaining that in earlier years "it was a disaster trying to show too much at one time" and that now she was "more cautious in not trying to present too much."

These data strongly suggested that Ms. Reese's goal to avoid confusion during instruction worked against attending to recursive partitioning (line F1) and notations for recording each stage in a sequence of partitions (line E1). In particular, although Ms. Reese did not say so explicitly, the data suggested that from her perspective drawing tick marks from left to right was a more straightforward method for subdividing intervals. Ms. Reese's comments reflected a perspective in which teachers present mathematics to keep lessons simple rather than develop kernels of important ideas evidenced in students' thinking.

Ms. Reese's Perspective on Visualizing and Learning

Ms. Reese's three overarching goals of helping students understand sizes of fractions, developing algorithms, and avoiding confusion lent some insight into her instruction, but she might still have helped students develop recursive partitioning if she thought it important: Teachers do tackle topics they know students find difficult. Ms. Reese might have lent greater weight to recursive partitioning if she understood processes for drawing representations, and the underlying mental operations they require, to be important for developing understandings of fractions. Further analysis of Ms. Reese's lessons and interviews suggested, however, that she took fractions strips and number lines instead as temporary aides for visualizing "amounts."

Four days after her classroom discussion about 7/12 and 5/8 (line B1), we played the excerpt for Ms. Reese. Recall that she referred to, but did not use, pictures of fraction strips during the original discussion. When asked why she did not have students work with strips at this point in the lesson, Ms. Reese explained that she was caught off guard and was not sure what to do. The interviewer then asked her to expand on the role she saw for drawn pictures in teaching and learning.

G1. Reese: I see that some of the kids, well to see [fraction strips], to begin with, helps you implant that in your brain. And then after that you can visualize it or draw pictures or even just visualize it. I mean, maybe *it's from the strip, to the picture, to the visualize in the brain could be the progression to the algorithm.*

Ms. Reese's comments suggested that fraction strips serve as images for helping students visualize sizes of amounts and that students should internalize these images as a step toward numeric methods. Moreover, after working through *Bits and Pieces I*, Ms. Reese may have assumed that her students were visualizing fraction strips as they worked on tasks in *Bits and Pieces II*. Examining the full set of lessons, we found five other examples in which Ms. Reese responded to difficulties by asking students to visualize, but not draw, fraction strips. Thus, comments that Ms. Reese made during interviews seemed to reflect understandings that shaped her practice. Moreover, they raised the possibility that when adjusting the location of the 1 on her number lines, Ms. Reese concentrated more on the final image showing sizes of "amounts" than on processes for partitioning units she perceived to be fixed.

During this same interview, Ms. Reese also explained that over time students should rely on pictures less and less:

H1. Reese: I think that [drawn representations are] an aid for [students] to solve a problem, *if they need that strategy*. They can always draw it on their paper, and *I think for them having a variety of different ways that they can choose what's best for them.*

Ms. Reese's comments turned to state testing for a moment and then continued:

H2. Reese: Eventually they have, I mean, you know, when they go to high school, it's going to get harder and harder and harder, and *they're going to have to move on*. At this point, the age that they are in sixth grade, I think that it's whatever that they need to use to be successful, and if they want to pull their fraction strips out on the test, then go ahead.

When examining Ms. Reese's lessons, we found she made further comments consistent with the understandings that students should return to fraction strips when "confused," that drawn representations play a temporary role in learning, and that drawn representations should be replaced with numeric methods that provide more efficient paths to the same answers. In short, students should move from drawn to numeric methods quickly.[6]

Ms. Reese's Perspective on Multiple Solution Methods and Learning

Data presented to this point hint at the role that Ms. Reese saw for multiple problem-solving methods in learning: Students should have access to a variety of methods so that they can "choose what's best for them" (line H1). After reviewing the entire set of lessons, we found only two examples in which Ms. Reese discussed explicit connections between drawn and numeric methods. The first example occurred following the whole-class discussion about estimating 7/12 + 5/8 (line B1). Ms. Reese had students examine whether the computation 14/24 + 15/24 = 29/24 would make sense. She referred to, but did not draw, fraction strips when explaining that 7/12 is equivalent to 14/24 and that 5/8 is equivalent to 15/24. She reminded students that in previous activities they had had to make "the pieces the same size if [they] wanted to compare [fractions]." She concluded by asking, "How many pieces in all would you get if you combined those, and they're all 24th-strip pieces?" The second example occurred during partner work 2 weeks later when she had a student draw fraction strips to determine 1/2 + 1/3. In both examples, Ms. Reese expected students to generate equivalent fractions with (imagined) strips and then count the resulting pieces.

Although Ms. Reese evidenced connections between partitioning fraction strips and generating common denominators, comments that she made during her interview on April 8 suggested that she focused primarily on getting the same answers:

I1. Reese: I think if I were to ask them, "How . . . what would happen if I gave you this problem and asked you to do it on the number line, could you do that?" "Yes." "Well, what if I gave you that same problem and said use the rule that you came up with in class, *would you get the same answer*?" They would tell me, "Yes." Which means they understand they're the same.

Moreover, Ms. Reese explicitly separated drawn and numeric methods for adding and subtracting fractions during her interview on April 21:

J1. Reese: I want them to be able to, *be able to use the number line and be able to do the algorithm independently of each other*. We have been doing it all along both, number line, you know, algorithm, number line, algorithm. . . . And I think that it's good to do both, but I think *they should be able to do them independently as well so that, that I know that they truly understand both.*

These data demonstrated that Ms. Reese valued multiple methods because they provide alternatives from which each student can find at least one reliable method (line H1). She focused on the fact that number-line and numeric methods give the same answers (line I1) and, at times, intentionally kept number-line and numeric methods separate (line J1).

Taken together, Ms. Reese's goals and understandings appeared to support each other and to direct her attention toward partitioning from left to right. If students

[6] We did not have data to establish strong connections between Ms. Reese's experiences teaching high school and teaching 6th grade but wondered whether she associated continued use of drawn representations with low achievement because of her experiences using manipulatives with 11th-grade students who had failed algebra.

develop fraction algorithms by internalizing images of "amounts" composed of equal-sized pieces, then it makes sense to produce such images using a straightforward method. From Ms. Reese's perspective, partitioning from left to right was less likely to confuse students, but recursive partitioning did make sense in select situations in which accurate pictures mattered. She wanted all students to find and understand a reliable method, but devoting significant time to different drawn methods did not make sense when they were about to be replaced by numeric methods that produced the same answers more efficiently. Moreover, she perceived that her students were ready to transition to numeric methods.

Sonya's Interpretation of Number Lines When Benchmarks Are Estimated Locations

Like Ms. Reese, Sonya also used two methods for establishing partitioned units, one in which she drew tick marks from left to right and one in which she took partitions of partitions. This section analyzes cognitive structures that Sonya evidenced for establishing partitioned units, structures through which she probably interpreted the second step in the demonstrated pattern of solutions. The data below came after initial evidence for the students' *n*-out-of-*m* structure (Figure 2a) but before evidence that they assimilated Ms. Reese's instruction through the same structure (lines D1–D7). Thus, the analysis below of Sonya's further structures suggests that the *n*-out-of-*m* structure was one of several weakly connected structures that led to inconsistent reasoning about fractions and partitioning.

Data from Sonya's and Jenny's second interview, March 21, demonstrated that when Sonya attended to a fixed unit, she could take partitions of partitions—at least by taking halves of halves—and could place equivalent fractions at the same location. The interviewer was investigating the students' ability to represent equivalent fractions and asked them to draw halves and fourths strips. Sonya drew a halves strip and corrected Jenny when she drew a fourths strip with unequal pieces directly underneath (Figure 5a). Sonya said, "[Jenny] wanted the two fourths to line up with the half [mark]." Jenny revised the locations of her fourths as shown in Figure 5b. A few exchanges later, Sonya explained, "The one fourths, it should be half of the half, like right there" and pointed to the appropriate location for one fourth on her halves strip. She continued by saying that three fourths should be "right in the middle" between one half and one whole. Finally, the interviewer asked the students to draw a sixths strip. Sonya outlined a new strip underneath and the same size as the fourths strip. She pointed to two fourths, and brought her hand down to locate three sixths. She then subdivided each half with two approximately equally spaced tick marks (bottom strip, Figure 5b). When the interviewer asked if the strips could be different sizes, Sonya said, "They need to be the same size." Because Jenny had drawn the initial fourths strip, whether Sonya could establish a fixed unit on her own remained unclear.

During Sonya's and Jenny's third interview, March 28, Sonya used understandings of benchmarks as estimated locations that did not appear well connected with her other understandings described thus far. (Recall that estimation lessons at the

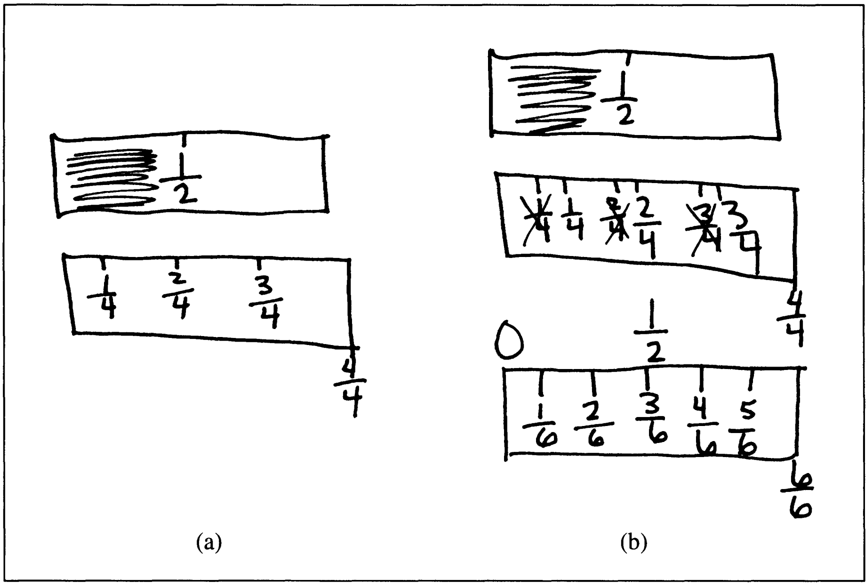

Figure 5. (a) Sonya's halves strip and Jenny's initial fourths strip. (b) Jenny's revised fourths strip. Original work in (a) and (b) has been retraced.

beginning of the unit introduce the use of 0, 1/2, and 1 as benchmarks or reference points, and that the first of the three steps in the pattern of demonstrated solutions also used whole numbers to establish reference points on the number line.) The interviewer asked the students to solve 2/3 + 3/4 on the number line. Sonya drew a number line and put "0" on the left-hand end, "1" in the middle, and "2" on the right-hand end.[7] Sonya then re-expressed 2/3 + 3/4 first as 4/6 + 6/8 and then as 8/12 + 12/16. Thus, she doubled numerators and denominators to generate correct equivalent fractions but did not make progress toward common denominators. When asked to use the denominator of 12, the students generated 8/12 and 9/12 quickly.

Sonya then put 12 tick marks, this time from left to right, between 0 and 1. Jenny did the same between 1 and 2. The students now had one too many tick marks in each unit interval. Sonya proceeded to label the tick marks, resulting in a number line that showed 12/12 and 1 in two separate locations. (Sonya's original 1 is the longer tick mark labeled 13/12 in Figure 6a.) The students also drew arrows but did not represent the sum correctly. When the interviewer asked about the locations of 1 and 12/12, Sonya pointed to the 1 when explaining "half of 2 is 1" and pointed to the 12/12 when explaining "the whole number is right there." Sonya also explained that 12/12 and 1 were the same because they were equal. The interviewer

[7] Why Sonya used more reasonable whole numbers here than in her subsequent April 9 interview (Figure 3) remained unclear.

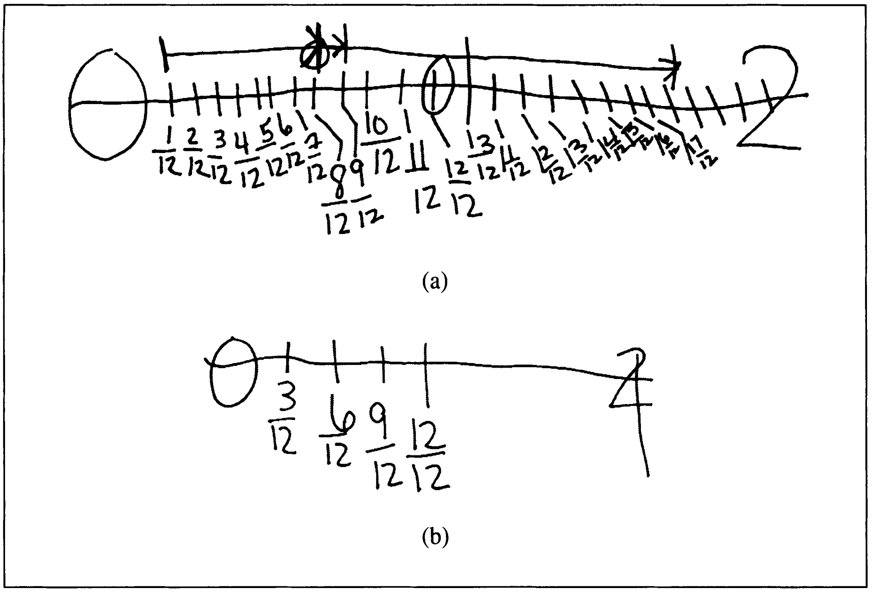

Figure 6. (a) Sonya's number line when benchmarks are estimated locations. (b) Sonya's number line when benchmarks are exact locations. Original work in (a) and (b) has been retraced.

asked, "Does it make a difference if you have two different places where you've marked the same thing?" Sonya replied, "No. Because both of them equals 1. When you draw your number line, you have to put 12 of them in each space because that is what the denominator is."

The interviewer then showed lesson video in which Ms. Reese demonstrated 1/4 + 1/8 on the number line (Figure 1). He asked the students what they noticed, and they mentioned the fractions 1/8, 2/8, and 3/8 and the two arrows. When the interviewer asked what else they noticed, Sonya said, "The whole number, instead of putting 8 over 8, she put 1." (During the lesson excerpt Ms. Reese asked where 8 eighths was on the number line, and several students commented, "On the 1.") A few exchanges later, Sonya explained that Ms. Reese would have put 8/8 "where the 1 is now" and pointed to Ms. Reese's number line appropriately. The interviewer returned the students' attention to their work and asked again about 12/12 and 1 being in two different places. Sonya explained:

K1. Sonya: You are trying, on the 0, 1, and 2 (pointed to "0," "1," and "2"), you are trying to put where 8 over 12 and 9 over 12, *you are trying to guess* where, *like a estimate*, you trying to put where it goes here, and then when I drew these (pointed to 12ths tick marks), when she drew these, you are, like, telling, *you know where 12 over 12 is at.*

Sonya had made a passing remark during the previous interview about benchmarks being estimates, but we did not attach much significance to the comment at the time. Here Sonya's discussion of 0, 1, and 2 as benchmarks suggested that she and Jenny placed them as estimates and then iterated a subinterval from left to right to determine the actual location of the 1. This approach was consistent with several other examples in which Sonya iterated a length to locate the whole, including her attempt to create a thirds strip by extending the halves strip in Figure 2b. Sonya's explanation was also strikingly similar to those cases in which Ms. Reese adjusted the location of the 1 but, instead of erasing the original 1, the students now had two 1's. Thus, for these students, benchmarks appeared not to establish fixed unit intervals.

Whether Sonya referred to tick marks or spaces when she said "you have to put 12 of them in each space" remained unclear. At earlier points, she sometimes connected the denominator with spaces or pieces (e.g., Figure 2a) but, at one point at the beginning of this interview, she clearly connected the denominator with the number of tick marks in an interval. Although Sonya placed equivalent fractions at one location and connected fractions of the form *n/n* with the whole at earlier moments during her interviews (e.g., Figure 5b), the fact that the 1 and 12/12 were in two different locations did not appear to create a perturbation for her.

When the interviewer asked the students to draw a new number line, thinking of benchmarks as exact locations, not estimates, Sonya used partitions of partitions once more. She drew a new number line with "0," "1," and "2" labels, added a "12/12" label under the "1," added a tick mark for 6/12, and explained that "half of 12 is 6" (Figure 6b). The interviewer asked where she would put "3 over 12," and Sonya replied "Right there (pointed halfway between 0 and 6/12) because half of 6 is 3." The interviewer asked where she would put "9 over 12," and Sonya replied:

L1. Sonya: Right there (wrote "9/12" halfway between 6/12 and 12/12) because it's in between, because 3, 6, 9. You gotta make half of 3 is 6 (pointed to 3/12 and then 6/12), and if you go up 3 more it'd be 9 (pointed to 9/12). So it is half (pointed to 12/12), so it is in between the 6. It is, like, it'd be 7 and then 8 (added tick marks) and then 9 (pointed to 9/12), and then 10, 11 (added tick marks). And you have 3 in between, I mean 2, 2 in between each one (pointed to two tick marks in each fourth).

She went on to locate 1 6/12 and 1 3/12, also appropriately.

As when discussing halves, fourths, and sixths strips (Figure 5), Sonya engaged understandings about partitions of partitions when she focused on a fixed unit. Moreover, reasoning about partitions of partitions ameliorated at least some of the difficulties she had representing fractions on number lines. Although Ms. Reese wanted to use a simple method, these data suggested that adding tick marks from left to right and moving the location of the 1, if necessary, unintentionally undermined opportunities for Sonya to engage and develop existing cognitive structures useful for reasoning about fractions as lengths.

Ms. Reese's Response to Sonya's Thinking

When we played excerpts from Sonya's interview, Ms. Reese was stumped by Sonya's discussion of 0, 1, and 2 as estimated locations. Ms. Reese commented, "She thought that maybe a number line could be an estimate, and I don't know why. We've never talked about a number line being an estimate." The interviewer recalled that Ms. Reese partitioned from left to right and pointed out that Sonya did something different when she treated the 1 as an exact location. Ms. Reese commented, "That's interesting how she did that," but did not elaborate. The interviewer then asked about making tick marks of different sizes, and line E1 is taken from Ms. Reese's response that emphasized avoiding confusion. Neither the interviewer nor Ms. Reese discussed possible connections between Sonya's understandings and Ms. Reese's practice of adjusting the location of the 1 if necessary.

DISCUSSION

The three-step pattern we observed (Figure 1) established a classroom mathematical practice in the sense that it contained a repeatedly observable pattern in solutions to fraction addition problems on number lines. Our analysis demonstrates that how drawings were produced contributed to miscommunication between Ms. Reese and Sonya. A coherent set of goals and understandings led Ms. Reese to partition what she perceived to be fixed units, most often from left to right, but the *n*-out-of-*m* structure and the understanding that benchmarks were estimates made it difficult for Sonya to focus on a fixed unit. When Sonya did perceive a fixed unit, however, she could begin to represent fractions appropriately by taking partitions of partitions. We do not claim that Ms. Reese would have ameliorated all of Sonya's difficulties by simply changing how she talked about "amounts" or by taking partitions of partitions more consistently. Sonya's difficulties regulating access to her knowledge base suggested that she would still have faced significant challenges with problem-solving aspects of CMP activities.

Although Ms. Reese and Sonya miscommunicated, students who evidenced more knowledge of fractions and linear measurement seemed to understand Ms. Reese's explanations more fully. Sam, the student Ms. Reese watched construct ninths from thirds when solving 1/3 + 2/9, was one of the students we interviewed. During lessons and interviews, he firmly maintained a fixed unit and the inverse relationship between the size and number of pieces in a partition. He also demonstrated two methods for partitioning, iterating a subinterval from left to right and taking partitions of partitions. When iterating from left to right, however, Sam would adjust the size of his subinterval until iteration exhausted exactly the established unit. Thus, he probably interpreted the second step in the demonstrated pattern of solutions differently than Sonya and in ways more compatible with Ms. Reese's interpretation.

We close with a discussion of how the present study could inform future research. First, we point out that key features of the present case may well be present in other classrooms. Further studies suggest that other students share some of Sonya's central difficulties with fractions. Studies mentioned in the background section above have reported that other students had trouble maintaining a fixed whole when interpreting fractions as pairs of whole numbers (e.g., Ball, 1993; Mack, 1990, 1993, 1995; Streefland, 1991). Furthermore, in one of the only other studies of which we are aware that has examined students' understandings of fractions on number lines, Bright, Behr, Post, and Wachsmuth (1988) reported that, like Sonya, fourth- and fifth-grade students had trouble connecting equivalent fractions with single points on the number line. The fact that many teachers are presently transitioning from more traditional to more reform-oriented materials suggests that others will face challenges similar to those experienced by Ms. Reese when using drawn representations to develop numeric methods (see Ball et al., 2001; Borko et al., 1992; and Ma, 1999 for further examples).

Second, results of the present study suggest that, with respect to the roles that representations can play in instruction, it would be profitable to elaborate current discussions of pedagogical content knowledge (e.g., Shulman, 1986) and mathematical knowledge for teaching (e.g., Ball & Bass, 2000) in at least two ways. Our results demonstrate that Ms. Reese underestimated the complexity of helping students like Sonya make substantive use of drawn methods for solving fraction arithmetic problems. A central issue was Ms. Reese's perspective that number-line methods serve as tools to visualize "amounts." She apparently did not attend to ways that drawn methods could be used to *deduce* numeric methods and, at times, made explicit decisions to keep the two separate (line J1). Other teachers with whom the CoSTAR project has worked have also treated drawn methods as alternative, separate paths to numeric answers. Although current discussions of pedagogical content knowledge and mathematical knowledge for teaching mention representations, we are not aware of further research that examines closely the different purposes for which teachers might use drawn methods.

Furthermore, although discussions of teacher knowledge have attended to representations (e.g., Ball, 1993; Ball & Bass, 2000; Shulman, 1986), little research has attended to teachers' interpretations and uses of fine-grained representational features such as tick marks. When discussions of pedagogical content knowledge and mathematical knowledge for teaching have turned to representations, they have not done so at a grain size that would distinguish between Ms. Reese's two partitioning methods. The present study demonstrated that seemingly subtle differences in how Ms. Reese established partitioned units had significant consequences for at least Sonya's opportunities to learn. More generally, discussions of pedagogical content knowledge and mathematical knowledge for teaching should treat resources teachers have for eliciting and adapting to students' interpretations and uses of drawn representations at a comparably fine grain size.

Third, the methods we developed for the present study had strengths and limitations. We do not claim to have identified all of the cognitive structures that Ms. Reese and Sonya used, but our analysis of talk, gesture, and inscription was sufficient to gain insight into incomplete communication that was not evident to us, and apparently not to them, from the lessons alone. At the same time, our methods reflect tradeoffs between gathering data sufficient for inferring teachers' and students' cognitive structures, which takes time, and for accessing teachers' and students' interpretations of particular classroom events, which can move quickly. A different research design might get closer to teachers' and students' interpretations of moment-to-moment classroom events by conducting interviews immediately after lessons. Such an approach would have different limitations. For instance, if a student paraphrased a teacher's explanation, it would be hard to determine from just those data whether the student imitated the teacher and masked weak understanding or interpreted the explanation through a well-developed knowledge base. We also struggled to anticipate which classroom and interview events would emerge as central during subsequent analysis. As a result, we did not interview Ms. Reese about connections between Sonya's *n*-out-of-*m* interpretation of fractions (e.g., lines D1-D7) and language that Ms. Reese used to explain the meaning of fractions (e.g., line B1). Finally, student and teacher interviews may have altered the course of subsequent lessons we observed. As one example, our data leave open the possibility that interview experiences helped Sam begin to construct recursive partitioning, which he then evidenced in class when solving 1/3 + 2/9. Continued development and refinement of methods for studying interactions between teachers' and students' interpretations of lessons in which they participate together is essential if we are to gain deeper insights into classroom teaching and learning and how they might be improved.

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