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Secondary School Mathematics Teachers' Conceptions of Proof

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Recent reform efforts call on secondary school mathematics teachers to provide *all* students with rich opportunities and experiences with proof *throughout* the secondary school mathematics curriculum—opportunities and experiences that reflect the nature and role of proof in the discipline of mathematics. Teachers' success in responding to this call, however, depends largely on their own conceptions of proof. This study examined 16 in-service secondary school mathematics teachers' conceptions of proof. Data were gathered from a series of interviews and teachers' written responses to researcher-designed tasks focusing on proof. The results of this study suggest that teachers recognize the variety of roles that proof plays in mathematics; noticeably absent, however, was a view of proof as a tool for learning mathematics. The results also suggest that many of the teachers hold limited views of the nature of proof in mathematics and demonstrated inadequate understandings of what constitutes proof.

Key Words: Proof; Secondary mathematics; Teacher beliefs; Teacher knowledge

Many consider proof to be central to the discipline of mathematics and the practice of mathematicians. In fact, Ross (1998) contended that "the essence of mathematics lies in proofs" (p. 254). Yet, surprisingly, the role of proof in secondary school mathematics has traditionally been peripheral at best, usually limited to the domain of Euclidean geometry. According to Wu (1996), however, the scarcity of proof outside of geometry is a misrepresentation of the nature of proof in mathematics. He argued that this absence is

> a glaring defect in the present-day mathematics education in high school, namely, the fact that outside geometry there are essentially no proofs. Even as anomalies in education go, this is certainly more anomalous than others inasmuch as it presents a totally falsified picture of mathematics itself (p. 228).

Similarly, Schoenfeld (1994) maintained that "proof is not a thing separable from mathematics, as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I believe it can be embedded in our curricula, at all levels" (p. 76). Many mathematicians and mathematics educators agree with Wu's and Schoenfeld's sentiments and, over the last 20 years, have been reassessing the nature and role of proof in mathematics educa-

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tion. This reassessment has influenced the practice of mathematicians, theories of mathematics education, and mathematics curricula (Hanna & Jahnke, 1993).

Reflecting this reassessment, as well as embracing the central role of proof in mathematics, recent reform efforts have significantly elevated the status of proof in school mathematics (National Council of Teachers of Mathematics [NCTM], 2000). In contrast to its conspicuous absence in previous recommendations (see NCTM, 1989), proof is expected to play a much more prominent role throughout the *entire* school mathematics curriculum and to be a part of the mathematics education of *all* students. Successfully enacting these new recommendations, however, places significant demands on school mathematics teachers because approaches designed to enhance the role of proof in the classroom require effort on their part (Chazan, 1990; Jones, 1997). The challenge of meeting these demands is particularly daunting in light of the fact that many students find the study of proof difficult (e.g., Balacheff, 1988; Bell, 1976; Chazan, 1993; Healy & Hoyles, 2000; Senk, 1985).

Factors that have been identified as important determinants of teachers' classroom practices, and that consequently have major implications for the extent to which teachers implement reform recommendations, are their subject matter knowledge and beliefs (Borko & Putnam, 1996). Accordingly, teachers' success in enhancing the role of proof in the classroom depends in large part on the nature of their own conceptions[1] of proof. Yet, to date, little research has focused on teachers' conceptions of proof and even less has examined *in-service secondary school* teachers' conceptions of proof—the focus of this study. Researchers have focused primarily on prospective elementary school (e.g., Martin & Harel, 1989; Simon & Blume, 1996) and prospective secondary school (e.g., Jones, 1997) teachers' conceptions of proof, as well as undergraduate mathematics majors' conceptions of proof (e.g., Harel & Sowder, 1998). Moreover, this body of research has tended to neglect individuals' views regarding the nature and role of proof, focusing instead on individuals' judgments of proof and approaches to proving. Consequently, with the increased emphasis on proof in school mathematics—in particular, in secondary school mathematics—as well as the accompanying demands on those currently teaching school mathematics, there exists a significant need for research on in-service secondary school mathematics teachers' conceptions of proof. A goal of this article is to describe results of a study that examined in-service secondary school mathematics teachers' conceptions of proof.

THEORETICAL PERSPECTIVES

Authors have suggested various roles that proof plays in mathematics: to verify that a statement is true, to explain why a statement is true, to communicate math-

[1] My use of the term *conceptions* includes both subject matter knowledge and beliefs. Although separating teachers' knowledge and beliefs serves as a useful heuristic for thinking about and studying factors influencing their instructional practices, the separation is less distinct in reality than it is in theory (Grossman, 1990).

ematical knowledge, to discover or create new mathematics, or to systematize statements into an axiomatic system (e.g., Bell, 1976; de Villiers, 1999; Hanna, 1983, 1990). These five roles compose the framework for considering teachers' conceptions of proof used in this paper. I will briefly elaborate on these roles.

The role of proof in verifying that a statement is true requires little elaboration. Indeed, few would question that a main role of proof in mathematics is to demonstrate the correctness of a result or the truth of a statement (Hanna, 1983). Yet, mathematicians expect the role of proof to include more than a simple verification of results—that is, according to Hersh (1993), mathematicians are interested in "more than *whether* a conjecture is correct, mathematicians want to know *why* it is correct" (p. 390). Moreover, a proof functioning in this latter role—explaining why a statement is true—is often held in higher regard: The status of a proof is enhanced if it gives insight as to why the proposition is true as opposed to just confirming that it is true (Bell, 1976).

Many within the mathematics community also view proof as "a form of discourse" (Wheeler, 1990, p. 3)—that is, as a means for communicating mathematics to other mathematicians (Alibert & Thomas, 1991; Balacheff, 1991). Proponents of this view have described, for example, the interactive process by which an argument becomes a proof as "a debating forum" (Davis, 1986, p. 352) and as "an essentially public activity" (Bell, 1976, p. 24). Similarly, Hanna (1990) noted that "the acceptance of a theorem by practising mathematicians is a social process" (p. 8).

Proof also plays an important role in the discovery or creation of new mathematics. As de Villiers (1999) noted, "There are numerous examples in the history of mathematics where new results were discovered or invented in a purely deductive manner [e.g., non-Euclidean geometries]" (p. 5). In addition, this role of proof is manifest in the relationship of proof to problem solving and conjecturing (Pólya, 1957). Finally, the role of proof that is perhaps the "most characteristically mathematical" (Bell, 1976, p. 24) is its role in the systematization of results into a deductive system of definitions, axioms, and theorems.

In sum, an informed conception of proof—one that reflects the essence of proving in mathematical practice—must include a consideration of proof in each of these roles. Traditionally, however, there has been a long distance between these roles and their manifestation in school mathematics practices (Balacheff, 1991). In large part because of such inconsistencies, current reform efforts are calling for significant changes in the role of proof in school mathematics (NCTM, 2000; Ross, 1998). The goal of the study reported in this article was to examine a factor that is critical to the successful enactment of such recommendations—namely, teachers' conceptions of proof. The study was guided by the following questions: (a) What are teachers' conceptions about the role of proof? (b) What constitutes proof for teachers? and (c) What do teachers find convincing?

METHOD

Participants

Sixteen in-service secondary school (Grades 9–12) mathematics teachers participated in this study. Their years of teaching experience varied from 3 to 20 years, and the courses they taught ranged from pre-algebra to Advanced Placement (AP) Calculus. Eleven teachers had undergraduate mathematics degrees and 5 had undergraduate engineering or physical science degrees; 13 teachers also had master's degrees, two of which were in mathematics. The teachers used diverse curricular materials in their classrooms; some of them used reform-based curricular programs, and others used more traditional curricular programs.

The teachers were selected on the basis of their willingness to participate in the study and were chosen from among participants in two ongoing professional development programs. Although one might question how representative the participating teachers were of the larger population of secondary school mathematics teachers, it is worth noting that the participating teachers were committed to reform in mathematics education (as evidenced in part by their seeking professional development opportunities focusing on reform). Consequently, it is likely that these teachers were not only familiar with the most recent reform documents (e.g., NCTM, 2000) and the corresponding recommendations, but were also interested in changing their instructional practices to more closely reflect the vision of practice set forth in such documents.

Data Collection

The primary source of data was semistructured interviews. The data were collected in two distinct stages, each with its own primary focus. The first stage focused on teachers' conceptions of proof in the discipline of mathematics (i.e., teachers' conceptions as individuals who are knowledgeable about mathematics), whereas the second stage focused primarily on their conceptions of proof in the context of secondary school mathematics (i.e., teachers' conceptions as individuals who are teachers of secondary school mathematics).[2] The second stage of data collection was shaped by an initial analysis of data from the first stage and, as a result, the second stage also included follow-up tasks and interview questions pertaining to teachers' conceptions of proof in the discipline of mathematics. Because the focus of this article is on teachers' conceptions of proof in the discipline of mathematics, the results presented and subsequent discussion focus exclusively on data from the first stage and the relevant data from the second stage (see

[2] At times, this separation into two stages seemed somewhat artificial because the teachers often had trouble removing their "teacher hats" (i.e., the teachers' responses often reflected what they thought their students might do or think). Yet, I tried to maintain this separation throughout the data collection stages by reminding teachers to think about a question or task as someone who knows mathematics rather than as someone who teaches mathematics.

Knuth, 2002, for a discussion of teachers' conceptions of proof in the context of secondary school mathematics).

Each interview consisted of several parts. Initial questions focused on teachers' conceptions about the nature and role of proof in mathematics. Typical questions included: What does the notion proof mean to you? What does it mean to prove something? What purpose does proof serve in mathematics? How does an argument become a proof? Do proofs ever become invalid? Other questions focused on teachers' understandings of what constitutes proof. More specifically, during the interview, teachers were shown and asked to evaluate different sets of researcher-constructed arguments—arguments that varied in terms of their validity as proofs (cf. Martin & Harel, 1989; see Figure 1 for an example of three such arguments within a set). The arguments presented were chosen so that the underlying mathematical concepts were not difficult; ideally, the focus of the teachers would be on the argument presented rather than on trying to understand the mathematics needed to produce the argument.

The argument sets also provided a context for examining the nature of what teachers find convincing; in particular, teachers were asked whether they found a particular argument within a set more convincing than others, and if so, why. This additional task was included in an attempt to discern whether teachers were cognizant of the explanatory role of proof. To this end, the arguments in each set differed, to varying degrees, in the extent to which they were explanatory (cf. Hanna, 1990)—that is, the extent to which they provided "a set of reasons that derive from the phenomenon itself" (p. 9). Three arguments demonstrating this construct are displayed in Figure 1: Argument (a) provides little insight into why the statement is true, only that it is true; in contrast, Arguments (b) and (c) provide insight based on the geometric and algebraic representations, respectively, into why the statement is true. Although the explanatory variance of the arguments in each set required an a priori categorization, I hypothesized that the rationale that the teachers provided for their responses might provide an indication of the degree to which they found particular arguments more or less explanatory than other arguments within a set.

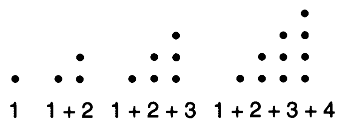
Data Analysis

The data analysis was grounded in an analytical-inductive method in which teacher responses were coded with external and internal codes and then classified according to relevant themes. Coding of the data began with a set of external (researcher-generated) codes that were identified prior to the data collection and that corresponded to, and were derived from, the theoretical framework (e.g., verification role of proof, explanation role of proof). The deductive approach used in producing the external codes was then supplemented with a more inductive approach (Spradley, 1979). As the data were being examined, emerging themes required the proposal of several new codes (e.g., sufficient detail as a criteria used in evaluating arguments, counterexamples may exist for a proof). After proposing

Prove: The sum of the first n positive integers is $n(n + 1)/2$.

(a) For $n = 1$ it is true, since $1 = 1(1 + 1)/2$.
Assume it is true for some arbitrary k, that is, $S(k) = k(k + 1)/2$.
Then consider:
$S(k + 1) = S(k) + (k + 1) = k(k + 1)/2 + k + 1 = (k + 1)(k + 2)/2$.
Therefore the statement is true for $k + 1$ if it is true for k.
By induction, the statement is true for all n.

(b) We can represent the sum of the first n positive integers as triangular numbers.



The dots form isosceles right triangles with the nth triangle containing:

$S(n) = 1 + 2 + 3 + 4 + ... + n$ dots.

Overlaying a second isosceles right triangle of the same size so that the diagonals coincide produces a square containing n^2 dots plus n extra dots due to the overlapping diagonals. To illustrate, the figure below represents the fourth isosceles right triangle and another of the same size overlaid so that the diagonals coincide. In this case, a square containing 4^2 dots plus 4 extra dots due to the overlapping diagonals is produced:

Therefore, in the general case (using the nth triangle), the number of dots produced by the two overlapping triangles is $2S(n) = n^2 + n$,
so $S(n) = (n^2 + n)/2$.

(c)
$$\begin{aligned} S(n) &= 1 + 2 + 3 + ... + n \\ S(n) &= n + (n - 1) + (n - 2) + ... + 1 \end{aligned}$$
Taking the sum of these two rows:
$$\begin{aligned} 2S(n) &= (1 + n) + [2 + (n - 1)] + [3 + (n - 2)] + ... + (n + 1) \\ &= (n + 1) + (n + 1) + (n + 1) + ... + (n + 1) \\ &= n(n + 1) \end{aligned}$$
Therefore, $S(n) = n(n + 1)/2$

Figure 1. Three arguments justifying the statement that the sum of the first n positive integers is $n(n + 1)/2$. From "Some Pedagogical Aspects of Proof," by G. Hanna, 1990, *Interchange*, *21*(1), pp. 10–11. Copyright 1990 by Kluwer Academic Press.

these internal (data-grounded) codes, the data for each individual teacher were then reexamined and recoded to incorporate these new codes. In addition, data for an individual teacher were examined for consistencies and inconsistencies in the nature of his or her responses; such consistencies/inconsistencies for individual teachers were then examined across data sets for all of the teachers, with a focus on themes among the consistencies/inconsistencies. Finally, as a means of checking the reliability of the coding and the appropriateness of the coding scheme, a second researcher read and coded samples of the interview transcripts. The coded samples from both researchers were then compared, and differences were discussed until resolved. Data were then recoded to take into account any changes made to the coding scheme.

On completion of the coding, a domain analysis of the data sets was conducted as a means of identifying, organizing, and understanding the relationships among the primary themes that emerged through the coding process (Spradley, 1979). According to Spradley, domains are categories of meanings that comprise smaller categories, which are linked to the corresponding domain by a single semantic relationship. Domains selected for this stage of the analysis were determined by the research questions—that is, the issues that were deemed important for this study provided a backdrop against which specific categories were proposed as the data sets were examined. For example, I used domain analysis techniques to identify the nature of what the teachers found convincing, and this process enabled me to identify characteristics of convincing arguments. In this case, the domain chosen was "convincing arguments," and the smaller categories were the particular characteristics ("characteristics of" being the semantic relationship linking the smaller categories to the domain). As in the approach taken in coding the data, a more inductive approach supplemented this deductive approach and led to the proposal of additional categories.

RESULTS

This section presents the results of the study and is organized by the three guiding research questions. Although data were collected from each individual teacher, in reporting the results, themes related to teachers' conceptions of proof are reported for the group rather than for individual teachers. Included in this presentation are frequency counts for the relevant themes as well as representative excerpts from the interviews (followed by teacher initials, which are pseudonyms). Only themes evident in the responses of at least 4 of the 16 teachers are presented.

What Were the Teachers' Conceptions of the Role of Proof?

As shown in Table 1, several themes were evident in teachers' responses to the interview questions. In this case, the themes corresponded well to the roles of proof proposed previously. Several new themes did emerge from the data, however, and these themes related specifically to the context of secondary school mathematics rather than to the discipline of mathematics.

Table 1
Roles of Proof in Mathematics Identified by Teachers

Roles of proof	Number of teachers
Establishment of truth	
Q: Do proofs ever become 'invalid'?	
proof not subject to contradictory evidence	4
proof dependent on axiomatic system	6
contradictory evidence possible	6
Q: Counterexample possible for a given proof (see Figure 2)?	
conclusion tested empirically prior to responding	4
hesitancy in responding	5
atypical case needs to be tested	5
Explanation	
Promoting understanding	0
Answering why	3
Communication of mathematics	12
Creation of knowledge/Systematization of results	8

Proof as a means of verification. All the teachers suggested that a primary role of proof in mathematics was to establish the truth of a statement, although they talked about the means by which truth is established in one of two ways. On the one hand, 11 teachers stated, to varying degrees, that truth is established by means of a logical or deductive argument. The following 2 teachers' responses are representative:

> I think it means to show logically that a certain statement or certain conjecture is true using theorems, logic, and going step by step. (KK)

> I see it as a logical argument that proves the conclusion. You're given a statement, and the logical argument has this statement as its conclusion. (SP)

On the other hand, 5 teachers used more general terms, suggesting that truth is established by means of a convincing argument. For example, one teacher stated that proof is "a convincing argument showing that something that is said to be true is actually true" (KA). Another teacher within this latter group recalled how her teaching experiences with reform-based curricula influenced her view of proof: "Having taught Discovering Geometry and really looking at the Interactive Mathematics Program and all the writing that they do in there, I guess now I'd say proof is really a convincing argument" (DL).

One of the powerful features of this role of proof concerns the generality of the conclusion—that is, the fact that a proof establishes the truth of a statement for all situations that satisfy the given conditions. Although all the teachers indicated that they viewed proof as a means of establishing the truth of a conclusion, they seemed to have varied understandings of the generality of a proof's conclusion, both at a general level and at a more specific level. In responding to a question that probed the fallibility of proofs (i.e., Do proofs ever become invalid?), four teachers commented that a proof is a proof and is not subject to contradictory evidence. Six other teachers demonstrated perhaps a more sophisticated understanding in that they recognized that a change in the axiomatic system for which a proof was constructed might render an argument invalid in the new system. One teacher explained her perspective in the following manner:

> I would say that once it's been proved, unless you introduce some new model, like in geometry you can prove [sic] that parallel lines never touch until you get on a sphere and then you have a whole different way of looking at something. So within the same context, I wouldn't think that too many things can change. (KA)

Similarly, another teacher stated that, "there's always a chance that something new will come along, like Euclidean versus non-Euclidean geometry. If the parameters are changed, or some new insight occurs, then a proof might no longer be valid" (KU). Somewhat surprisingly, given their mathematics background, the six remaining teachers responded that it might be possible to find a counterexample or some other form of contradictory evidence, thus rendering a proof "invalid." The following responses were typical:

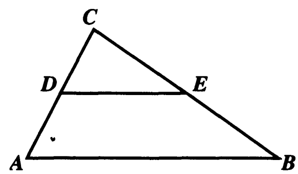
> Somebody could finally come up with a counterexample that proved it wrong. (QK)

> If it can be disproven by a counterexample.... Once it's proved, the probability may be there for a counterexample. (CC)

As will be discussed next, many of these teachers may not have a robust understanding of the generality of a proof.

The following discussion looks at the teachers' understandings of the fallible (or infallible) nature of proof more explicitly. The teachers demonstrated whether or not they had a robust understanding of a proof's generality by examining the relatively simple proof shown in Figure 2, telling if they understood it, and then answering questions about the conditions under which the conclusion would hold. All the teachers stated that the proof made sense and that they understood it. Next, they were asked if it was possible to find a counterexample.[3] Although every teacher correctly responded that it was not, four teachers drew additional triangles as a means of verifying for themselves the argument's conclusion prior to responding. One teacher who was asked why she bothered to test three additional triangles explained, "Because proof by exhaustion. There are millions of triangles that exist, and I've only looked at three" (SP). In addition, three other teachers did not respond very convincingly when asked if it was possible to find a counter-

[3] It is certainly possible teachers perceived the question as a "trick" question and, as a result, decided to verify the conclusion. In light of my relationship with the teachers (I was associated with a professional development project in which many of the teachers participated), however, I do not think this was the case.



Given: $\triangle ABC$ and points D and E, which are midpoints of AC and BC, respectively.
Prove: AB is parallel to DE.

D is midpoint of AC and E is midpoint of BC	Given.
$DC = (1/2)AC$ and $EC = (1/2)BC$	Definition of a midpoint.
$\angle C \cong \angle C$	Reflexive property.
$\triangle ABC \sim \triangle DEC$	If two sides of a triangle are proportional to the two corresponding sides of another triangle, and the included angles are congruent, then the triangles are similar.
$\angle CDE \cong \angle CAB$	Definition of similar triangles.
AB is parallel to DE	If two lines cut by a transversal (AC) form congruent angles with the transversal, then the lines are parallel.

Figure 2. Does the conclusion holds for all triangles? From "High School Geometry Students' Justification for Their Views of Empirical Evidence and Mathematical Proof," by D. Chazan, 1993, *Educational Studies in Mathematics*, *24*, p. 366. Copyright 1993 by Kluwer Academic Press.

example; they hesitated and then offered comments such as "I don't think so" or "I am pretty sure it is not possible."

In a second question related to this proof, I drew an atypical triangle (i.e., an "extremely" obtuse triangle with a very short base) and asked teachers if the conclusion would hold for this triangle. Again, every teacher correctly responded that the conclusion would still hold; however, five teachers actually made sketches of the given conditions and conclusion on the figure prior to responding (two of these five teachers also checked additional triangles in response to the previous question). One of these teachers drew another triangle—attempting to draw an even more atypical one—before stating that she thought the conclusion would hold:

> I'm thinking I want to see it a little more dramatically than you've drawn it, so I can make that decision. [*After drawing her own triangle:*] Yes, I think it will be true. (PB)

Four teachers also sounded somewhat unconfident (e.g., "I don't think so" or "It appears to be true"); two of these teachers had also expressed some degree of hesitancy in responding to the previous question. The level of conviction (or lack thereof) displayed by these teachers was quite different from that displayed by those teachers who had no doubt about the generality of this proof. As one of the confident teachers commented,

> That's the whole idea of doing a proof, it applies to every case. So of course it's going to be true. (KB)

A follow-up question asked teachers to explain the proof's generality (i.e., Why isn't the proof a proof only for the given triangle?). Every teacher, including those who expressed some doubt (either by testing examples or by responding unconfidently), provided satisfactory responses. It appears that the teachers who expressed doubt did not entirely believe what they purported to know. The following are representative of the teachers' explanations:

> There is nothing in the proof that uses any specialness about the way the triangle is drawn. It only uses the given words and this is just a picture to help us see what these words are talking about (KB).

> There is nothing about the particular figure that makes it unique, no measurements of sides or angles (KD).

One teacher, however, after explaining that it was just a general triangle and that the proof was based on its generic features, began to have second thoughts: "Now it makes me wonder if it would be true for all those special cases. I think it is" (EN).

In sum, the teachers expressed a view of proof as a means of establishing the truth of a statement, yet several teachers genuinely did not seem to understand (or, at the very least, did not seem to be confident in their understanding of) the generality of a proven statement (see Table 1). These teachers either believed that it might be possible to find some form of contradictory evidence to refute a proof or they expressed doubt regarding the conclusion of an argument even though they believed the argument to be a proof.

Proof as a means of explanation. There was no supporting evidence to suggest that the teachers viewed the promotion of understanding or insight as a role of proof in mathematics, in contrast to views espoused by many mathematicians (e.g., Hanna, 1990; Hersh, 1993). (That said, a caveat is in order: Three teachers did talk about the role of proof in explaining why something is true, and an additional seven teachers also mentioned this role of proof in secondary school mathematics.) Teacher responses included the following:

> I think of it [proof] as also answering the question of why does this work. (KA)

> A proof puts the "why" as to why we do something in a given situation. You can always go back to a proof to show why. (CA)

On the surface, these teachers' comments suggest that they do indeed view explanation as a role of proof; however, their comments pertained more to understanding how one proceeded from the premise to the conclusion of a proof—a procedural focus—rather than to understanding the underlying mathematical relationships illuminated by the proof. For example, these teachers viewed a derivation of the quadratic formula as an illustrative example of the role of proof in telling why something is true. One could follow the progression of steps in the derivation to understand how the general formula was produced (i.e., "why" it was true). The nature of the understanding connected with the role of proof in answering why, however, seems qualitatively different from the nature of the understanding connected with the role of proof in promoting insight. In the former role, a proof "shows only that a theorem is true," whereas in the latter role, a proof "shows why a theorem is true" (Hanna, 1990, p. 9).

Proof as a means of communicating mathematics. Twelve teachers expressed the view that proof arises from, or is a product of, social interaction. In the words of several of the teachers, proofs are a means to communicate and convince others of one's claims:

> [Proofs are a method] to convince them [i.e., a wider audience] that your thinking is correct. (DL)

> [Proofs are] an act of communication, for sure. Either writing down, or it could maybe be in spoken form through pictures and language, some logical sequence which convinces a reader or an audience that you have shown something must be true. (KB)

When asked how an argument becomes a proof, these teachers suggested that proof is the result of a particular social act—the acceptance of an argument by others:

> A mathematical community or audience for that particular proof would either accept it or refute it. (KJ)

> I think there has to be a collaboration between the prover and audience. "This is my argument and I believe my argument and this is my substantiation." A person who is trying to understand, if they disagree, then there has to be some interchange about accepting it or refuting it. (CA)

Proof as a means of creating knowledge and as a means of systematization. Eight teachers seemed to express the view that proof plays an essential role in the creation of mathematical knowledge and, to a lesser extent, in its systematization. These two roles are presented together because the teachers' responses included aspects of both roles. Representative excerpts from teachers expressing this view included the following:

> It's [i.e., proof is] the whole basis of our mathematics system. Everything is based upon being proven. (KK)

> We can start with something we know. We can go to something we don't know and [after constructing a proof] add that to the system. And then if we start with what we know we can add other things to this system by showing that it logically follows. (MQ)

> Math is a building block. Everything is based upon what was proven before. (CC)

Although the teachers' comments suggest an awareness of proof's role in creating knowledge—knowledge that subsequently becomes part of a larger "system" of knowledge—it is less clear whether these same teachers view such systematization in terms of some underlying axiomatic structure. In other words, it is less clear whether the teachers view the knowledge created by proofs as part of a deductive system of definitions, axioms, and theorems.

What Constitutes Proof for Teachers?

The teachers were given five sets of statements with 3 to 5 corresponding arguments justifying each statement; in all, there were 13 arguments that constituted proofs and 8 arguments that did not (each set of arguments had at least one nonproof). The arguments also varied in terms of the approach used in constructing them (e.g., algebraic, proof by induction) and their explanatory nature (i.e., more explanatory or less explanatory). Teachers were asked to review the various arguments within a set, to rate each argument in terms of its validity using a 4-point scale (cf. Martin & Harel, 1989), where 1 represented an argument that is not a proof and 4 represented an argument that is a proof, and to provide a rationale for the rating given to each argument. Ratings of 2 or 3 were included on the scale to allow teachers the freedom to express alternative views regarding the validity of a particular argument or to express varying degrees of certainty or acceptance regarding the validity of that particular argument. Further, employing a 4-point scale (as opposed to a 2-point scale) allows subtle facets of the notion of validity in proving to emerge—such as, for example, the presumption of assumed truths or the completeness of details. Prior to discussing the criteria that teachers used to evaluate the arguments, I will present the results of their evaluations.

Evaluation of arguments. In general, the teachers were successful in identifying those arguments that were proofs; over 93% of the ratings given to the arguments that constituted proofs were correct. Although it is impressive that teachers considered all the "right" arguments to be proofs, the number of nonproofs that they also rated as proofs was somewhat surprising, in light of their mathematics backgrounds. Overall, a third of the ratings that the teachers gave to the nonproofs were ratings as proofs. In fact, every teacher rated at least one of the eight nonproofs as a proof, and 11 teachers rated more than one as a proof. For example, 5 teachers rated the empirically based argument in Figure 3a as a proof. One teacher did not see it as a "formal" proof but as a proof, nonetheless: "It's a valid proof. It's not a formal proof, but it's still a proof of it" (LV). Two other teachers, however, were not quite as convinced of its validity; they found fault with it for reasons other than that the argument was based on empirical evidence. In the words of one, "They didn't try it for every type of triangle. [If they did] then I'd give it a 4" (KU). Apparently, if the faults were rectified, then the argument would be considered to be a proof.

(a) I tore up the angles of the obtuse triangle and put them together (as shown below).

The angles came together as a straight line, which is 180°. I also tried it for an acute triangle as well as a right triangle and the same thing happened. Therefore, the sum of the measures of the interior angles of a triangle is equal to 180°.

(b) Using the diagram below, imagine moving *BA* and *CA* to the perpendicular positions *BA'* and *CA"*, thus forming the second figure. In reversing this procedure (e.g., moving *BA'* back to *BA*), the amount of the right angle, *A'BC*, that is lost is *x*. This lost amount, however, is gained with angle *y* (*DA* is perpendicular to *BC*). A similar argument can be made for the other case. Thus, the sum of the measures of the interior angles of any triangle is equal to 180°.

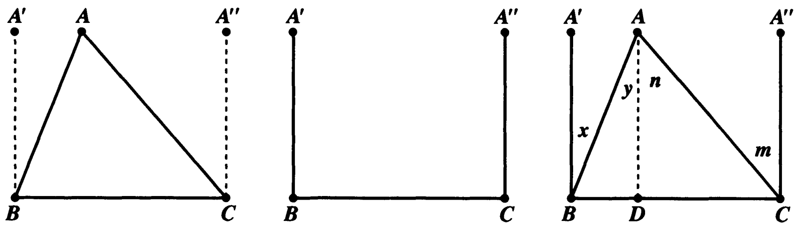


Figure 3. Two attempts at proving that the sum of the measures of the interior angles of any triangle is equal to 180°.

Note. Part (b) from "Students Proof Schemes: Results from Exploratory Studies" by G. Harel and L. Sowder in *Research in Collegiate Mathematics Education III* (p. 259), edited by A. Schoenfeld, J. Kaput, and E. Dubinsky, 1998, Washington, DC: Mathematics Association of America (MAA). Copyright 1998 by MAA.

For a second example, 10 teachers considered the argument displayed in Figure 4a to be a proof, though the argument actually proves the converse of the given statement. In determining the argument's validity, these teachers seemed to focus solely on the correctness of the algebraic manipulations rather than on the mathematical validity of the argument.

The algebraic steps were easy to follow, and I had no problem with it. (NA)

That proves it. It's just the algebra. Everything fits. (QK)

I'm convinced by [A; shown in Figure 4a]. A is a 4. I think what's convincing to me is that it's gone through the algebraic manipulation to show it. (PB)

Prove: If $x > 0$, then $x + \frac{1}{x} \geq 2$.

(a) $x + \frac{1}{x} \geq 2$ — Assume true.

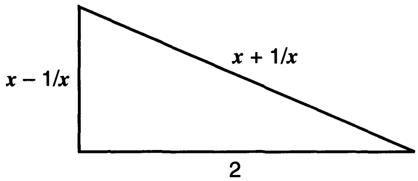
$\frac{x^2 - 2x + 1}{x} \geq 0$ — Subtracting 2 from each side and rewriting the left-hand side as a fraction with denominator of x.

$\frac{(x-1)^2}{x} \geq 0$ — Factoring the numerator.

$x > 0$ — Since the numerator is always positive and the fraction itself must be greater than or equal to zero, then the denominator must be positive.

Therefore, $x + \frac{1}{x} \geq 2$ is equivalent to $x > 0$. It follows that if $x > 0$, then $x + \frac{1}{x} \geq 2$.

(b) We can construct a right triangle with the given sides so that it satisfies the Pythagorean Theorem.



Note: If $0 < x < 1$ then the vertical side has length $\frac{1}{x} - x$.

That is, the following is a true statement: $\left(x - \frac{1}{x}\right)^2 + 2^2 = \left(x + \frac{1}{x}\right)^2$

From right triangle geometry, we know that the hypotenuse is longer than either leg.

Thus, $x + \frac{1}{x} \geq 2$.

Figure 4. Two attempts at proving that $x > 0$, then $x + 1/x \geq 2$: (a) a proof of the converse and (b) a proof using a visual representation. From "On Proofs and Their Performance as Works of Art," by G. Winicki-Landman, 1998, *Mathematics Teacher*, *91*, pp. 722–723. Copyright 1998 by the National Council of Teachers of Mathematics.

> A is just an algebraic approach.... Algebraic manipulation is basically the tool pulling this proof together (CA).

It is possible that, in the context of a formal interview, teachers may not have reviewed the argument in sufficient detail; nevertheless, their responses seem to suggest that they focused on the correctness of the manipulations performed in the argument as opposed to the nature of the argument itself. Of those teachers who did recognize the argument as a proof of the converse, the following response was typical: "I'm having trouble with A because you're starting with the assumption that the conclusion is true. I'm really struggling with this as a proof" (DL). It is interesting that this teacher still seemed to lack the confidence (or understanding) to conclude outright that the argument was not a proof of the given statement.

For a final example, four teachers rated the proof of the particular case shown in Figure 5 as a proof of the general case. Six teachers rated the proof of the particular case as a proof of the particular case only, and the remaining six teachers indicated that the proof of the particular case was not a proof because it was not general. Of those teachers who felt that the proof of the particular case was a proof of the general case, the following was a typical reason: "I gave it a 4 because even though it used a specific example and not a general case, it was clear from that example that this statement was true" (SP). On the one hand, these teachers may have abstracted the general argument from this proof—in a sense, mentally filling in the blanks that would be necessary to generalize it—a relatively easy task. As one teacher pointed out,

> The idea is in A [proof of the particular case]. All you have to do is go put x, y, and z in for 7, 5, and 6. You'd have the same proof and then it'd be a 4. (CA)

On the other hand, these teachers may have perceived the proof of the particular as more convincing (as will be discussed shortly) and thus accepted it as a proof.

Criteria used in evaluating arguments. Although teachers provided a variety of criteria in determining what constituted a proof and often supplied different criteria

Prove: If the sum of the digits of a whole number is divisible by 3, then the number itself is divisible by 3.

Consider 756. This number can be represented as follows:

$$756 = 7 \cdot 100 + 5 \cdot 10 + 6.$$

By the commutative and associative properties, we get

$$756 = (7 \cdot 99 + 7) + (5 \cdot 9 + 5) + 6 = (7 \cdot 99 + 5 \cdot 9) + (7 + 5 + 6).$$

Notice that the expression $7 \cdot 99 + 5 \cdot 9$ is always divisible by 9, and therefore also by 3. Now, since the second expression $7 + 5 + 6$, which is the sum of the number's digits, is divisible by 3, then, by the "sum property," we get that the number itself is divisible by 3. Since any number can be expressed in a similar fashion, then for any whole number, if the sum of its digits is divisible by 3, then the number itself is divisible by 3.

Figure 5. A proof of the particular as a proof of the general. From "Proof Frames of Preservice Elementary Teachers," by W. G. Martin and G. Harel, 1989, *Journal for Research in Mathematics Education, 20*, pp. 43, 45. Copyright 1989 by the National Council of Teachers of Mathematics.

for different arguments, they used four criteria more often than others. The first two criteria—*valid methods* and *mathematically sound*—primarily concerned qualities that teachers perceived as necessary in order for arguments even to be considered as proofs. In this case, teachers' ratings of arguments as proofs were based primarily on the particular proving method used or the mathematical reasoning involved in presenting the argument. The last two criteria—*sufficient detail* and *knowledge dependent*—primarily concerned distinctions that teachers made among arguments that they considered to be proofs. Once the issue of validity was resolved (i.e., an argument was identified as a proof), teachers then seemed to focus on qualitative factors in the presentation of the argument. In particular, they gave scores of 3 to arguments not quite meeting their own standards of proof but which they nevertheless considered to be proofs. Especially striking was the number of teachers who assigned a rating of 3 to arguments that they in fact considered to be proofs—in essence, proposing "degrees of completeness" that affected their rating of a proof. In short, the teachers used mathematically grounded criteria for accepting an argument as a proof while using qualitatively grounded criteria for making distinctions among proofs.

In all, seven teachers used the *valid methods* criterion in determining whether an argument was a proof. The focus of the teachers who applied this criterion was primarily on the method (or perhaps the form) used in producing an argument rather than on the reasoning presented in it. For example, one teacher correctly rated several arguments within a set as proofs (see Figure 1a–c). However, she viewed one of these arguments as being "better" than the others because of the particular approach taken in constructing the argument:

> [The proof shown in Figure 1c] feels likes it's mathematically valid, because it's algebraic in nature. So I tend to see this as the best. (MQ)

Other teachers did not necessarily base their acceptance of an argument as a proof on an understanding of the argument but rather on "knowing" that the method or approach used in constructing the argument was valid. For example, one teacher, who confessed to not really understanding proof by induction (see Figure 1a), nevertheless found such a proof convincing because of its method: "I know that that is a valid way of proving things" (KA). Similarly, another teacher commented, "I know that this [proof by induction] is one I've seen used before, and I assume it's a good way to do it" (KJ). Thus, in both of these cases, the teachers were convinced that the argument was a proof because of the method employed rather than because of an understanding of the method itself. The teachers who rated the proof of the converse of a statement (see Figure 4a) as a proof of the statement also used this criterion. In this case, the teachers focused not on the reasoning presented in the argument but rather on the symbolic manipulations performed in arriving at the conclusion. The manipulations were correct; thus, the teachers considered the argument to be valid. As one teacher put it, "I think what's convincing to me is that it's gone through algebraic manipulation to show it" (PB). Apparently, these teachers focused on the argument's local characteristics (i.e., moving from one step to the next within the argument) at the expense of attending to more global characteristics (i.e., the argument as a whole). Finally, one teacher gave two arguments ratings of 3 even though she considered them to be proofs; she assigned the lower rating because they were not "mathematical" methods of proving (see Figure 1b for one of the two proofs). As she explained, the reason for the lower rating stemmed from her experiences as a university mathematics student:

> These are all good, but Dr. Smith [a mathematics professor] would not have accepted these.... I know from the guys that I had at school that these are cute, and they would say well, that's real nice and that's a real interesting way to approach it, but that's more like a high school, middle school approach. This is a mature mathematician right here [a proof by induction].... There's nothing the matter with any of them ... but this is the one that I learned was *the* way, from a mathematician's point of view. (DF)

Thirteen teachers based their determination of an argument's validity on whether the argument was *mathematically sound,* as opposed to focusing on the particular method or approach used. In other words, these teachers focused explicitly on the validity of the reasoning presented in an argument. For example, one teacher rated an argument as a proof because it established the truth of the general case mathematically: "They showed it geometrically for the first few cases and then explained the generalization, why the statement is true for any size or shape in the general case" (SP). Another teacher based her rating of an argument on her understanding of the underlying mathematics:

> I agree with what an ellipse is, a set of points whose distance to two points is a constant, and the only point which would lie both on the line and on the ellipse is that point of tangency which is going to be the midpoint of that segment, which is very obvious. That's convincing as a proof to me. (KB)

In short, teachers who applied this criterion spoke more about the mathematics underlying an argument than about the particular mathematical method or approach used in constructing an argument.

Twelve teachers required that an argument deemed to be a proof must have *sufficient detail* in order to merit the highest rating. To some extent, one might expect teachers to make this criterion essential—in their daily practices, teachers frequently request that students "show all the steps" in their work. Several teachers' comments about some of the arguments capture the gist of this criterion:

> There are general requirements of a proof, even though you may think you're proving something soundly, if someone else was to look at it and couldn't follow it, then one of the requirements of a proof is missing.... It is a sound proof [rated a 3], although it would have been nice to have had a little bit more explanation so that I could have followed what they were doing a little bit better. (KU)

> This one [shown in Figure 4b] is algebraic and I think if I actually went through and wrote the steps out it would be a 4 for me. But you skipped steps here [i.e., details regarding the simplification of the equality]. (CC)

> [The proof shown in Figure 3b] is interesting.... It's a little hard for me to see. It's more the way the statement is written. It's not whether it's a valid idea. That's why I'm rating it down a little bit, it's not obvious to me. (CA)

> I don't like the proof [shown in Figure 3b]. I had to read it three times before I could follow it. I'd say it's a valid proof [rated a 4] though. (DN)

> I can't find anything in the mathematics that's wrong. There's nothing in it that's wrong, for me it left too much to figure out. I'd go with a 3. (DL)

> It just seemed to be more clear, very easy to follow, I don't even think I would have to ask a question of this person. So I would say that's a 4. (EN)

It is clear that for these teachers to give an argument the highest rating, not only did it have to be mathematically sound but it also had to include enough detail so that one could easily follow it. Thus, the rationale for giving an argument a rating of 3—still enough to be considered a proof—was more qualitative than substantive in nature. In other words, in assigning a rating of 4, these teachers seemed to focus more on the completeness of the steps of the argument and the ease with which one could follow it than on the argument as a whole.

Five teachers determined an argument's validity on the basis of the final criterion: whether an argument was *knowledge dependent*—that is, whether specialized knowledge was necessary to understand an argument. For example, in explaining why she gave a calculus-based argument a rating of 3, one teacher referred to the level of mathematics knowledge required: "You have to have a certain amount of math knowledge, but some people who haven't seen calculus in a while would find it helpful to have a short little explanation about the derivative" (KU). Another teacher, after assigning an argument a 3, stated what she valued in a proof: "I'd like a proof that if you don't know anything, and you're coming in and you read it, and you understand from the proof how everything falls out" (LV). In a similar fashion, one teacher explained why she gave one argument a 4—"I think this would convince any crowd without having them have to assume anything or know anything"—while rating another one only a 3—"Just a 3 because there is always the assumption that everybody knows about the derivative" (EN). Thus, these teachers gave a higher rating to arguments that required a less sophisticated understanding of the mathematics presented (or, alternatively, presented less sophisticated mathematics in constructing the argument).

What Do Teachers Find Convincing?

For each set of arguments, the teachers were also asked if they were more convinced by a particular argument or arguments, and why. It is worth mentioning that the most convincing arguments, as indicated by the teachers, were often not proofs. Table 2 displays the characteristics of arguments that teachers found convincing. Some of these characteristics also emerged as criteria that teachers used in evaluating an argument's validity (e.g., the criterion *mathematically sound* was used both to judge the validity of an argument and to describe why an argument was more or most convincing), and these will not be discussed again here. Descriptions of the other characteristics follow.

Concreteness. The defining characteristic of arguments in this category was their inclusion of a concrete feature; that is, teachers were most convinced by arguments

Table 2
Characteristics of Convincing Arguments.

Characteristic	Frequency [a]
Concrete features (i.e., uses specific values or a visual representation)	15 (13)
Familiarity	15 (10)
Sufficient level of detail	12 (8)
Generality	10 (9)
Shows why	9 (5)
Valid method	5 (3)
All equally convincing (of arguments within a set)	4 (4)

[a] The frequency is the number of occurrences of a particular reason. Totals may include multiple counts for a single teacher (e.g., a teacher may have been most convinced by familiar arguments in more than one set of arguments). The number of different teachers citing a particular characteristic is provided in parentheses.

that relied on specific examples (6 teachers) or provided a visual reference (12 teachers). For instance, one teacher found an argument based on several examples to be the most convincing: "[Argument] C [not shown in this article] convinces me the most. Seeing that many examples" (LV). Others found the proof of the particular case (see Figure 5) to be the most convincing because it showed the statement to be true by use of a specific example. As one teacher commented, "I should be more convinced by C [a proof of the general case—not shown] because it's much more general, but I like the particular one" (QK). In other cases, teachers were most convinced by an argument's visual features. As one teacher stated, "The most convincing to me is this one [see Figure 3a] because you can see it. It's there. It's a straight line" (CC). Another teacher stated, "The one that is most convincing to me right now is [Figure 4b] because it's the easiest to follow. I can see the right triangle and how it relates to the formula" (SP). For this particular teacher, the picture allowed her to connect the algebraic manipulations to something concrete.

Familiarity. Ten teachers found a particular argument in a set to be most convincing because of its familiarity (i.e., they had previously seen it or had used it in their instructional practice). In this case, the explanation of why they were convinced by an argument was not based on the mathematics presented but rather on their previous experiences with the argument. The following statements are representative:

> This one [Figure 1c] is pretty convincing based on my past experiences. For me, the convincing part of this one is the understandability because of my past experiences. (PB)

> [It's convincing because] I'm teaching conic sections and the ellipse and the definitions, this is very familiar to me and it just clicked. (NA)

> I mean I'm very familiar with this type of proof. I've taught geometry for many years. (KU)

It is certainly possible that these teachers were convinced by an argument resting on mathematical grounds, yet in the context of the interviews, their descriptions of why an argument was convincing made no reference to the mathematics involved in producing it but only to their previous experiences with an argument.

Generality. Nine teachers were most convinced by arguments that clearly proved the general case—that is, arguments that established the truth of a statement for all relevant cases. For example, several teachers found the proof of the general case (see Figure 5 for the problem statement and a proof of the particular case) to be most convincing. The following comments capture the essence of these teachers' views:

> Because it generalized this problem completely. It showed it to be true for any number. (SP)

> I thought proofs A [see Figure 5] and C [proof of the general case; not shown] were essentially the same thing. Proof A just used a specific number and proof C was the generalization of that. (KA)

Shows why. Five teachers were more convinced by an argument because of the insight that it offered into the underlying mathematics. This characteristic allowed these teachers to see not only that the statement proved was true, but also why it was true. As one teacher stated,

> Some of them really show an insight and some of them don't. C [Figure 1b] would definitely convince me of the truth because it's very visual and it shows exactly why it's true. (MQ)

It is interesting that the only arguments that teachers identified as convincing because they offered an insight were arguments that included visual representations. This was the case despite the fact that the argument sets were designed to include other, nonvisual, arguments that were (thought to be) explanatory (see, for example, Figure 1c).

DISCUSSION

This article reports results from a study that examined the nature of secondary school mathematics teachers' conceptions of proof. This section discusses the results in relation to teachers' views of the role of proof in mathematics, what constitutes proof, and what they find convincing, as well as the implications of this work for teacher education and mathematics education research.

Roles of Proof

The teachers described a variety of roles that proof plays in mathematics: to verify the truth of a statement, to explain why a statement is true, to communicate mathematics, and to create and systematize mathematics. These roles suggest that teachers have a diverse and, pedagogically speaking, potentially powerful understanding of the function of proof in mathematics. Perhaps if teachers were to pay explicit attention to these roles during their instruction, they would provide classroom experiences with proof that would enable their students to go beyond the limited conceptions of proof that students have traditionally developed. For example, one might question whether high school geometry students are able to view the proofs that they construct in class as interrelated—that is, whether these students are cognizant of the particular axiomatic system (typically Euclidean geometry) that provides the structure for their work. Teachers holding a view of proof as a means of systematization might be more likely to provide opportunities for students to reflect on their work through this particular lens. At the very least, these teachers would be better prepared to provide such opportunities for their students (cf. Healy, 1993). Encouraging students to reflect on proof from a "meta-level" may help them develop an understanding of issues related to the creation and systematization roles of proof. A parallel argument could be made concerning undergraduate mathematics education: As undergraduates, do prospective teachers have opportunities to experience and discuss these roles of proof? The Moore method of teaching, for example, which is used by some mathematicians, provides undergraduate students with just such an experience.[4] In the context of Euclidean geometry, for instance, students are given a few axioms and then left to deduce the consequences; in the process, they are both creating mathematical knowledge through the proofs they construct as well as developing an understanding of a particular deductive system of definitions, axioms, and theorems.

Noticeably absent among the roles of proof that teachers mentioned, however—and perhaps most important pedagogically—was its role in promoting understanding. To some extent, this omission should not be surprising as the focus of teachers' previous experiences with proof as students themselves, both at the secondary and collegiate levels, has tended to be primarily on the deductive mechanism or on the final product (Chazan, 1993; Harel & Sowder, 1998). As a consequence, "in most instructional contexts proof has no personal meaning or explanatory power for students" (Schoenfeld, 1994, p. 75). Further, proving practices in secondary mathematics classrooms are often limited to verifying the truth of statements that students (and teachers) know have been proven before and, in many cases, are intuitively obvious to them. Proving practices of this nature not only constrain the conceptions that students develop but also may limit the conceptions that teachers develop (as teachers' instructional practices may influence their conceptions). In short, if teachers are to develop a view of proof as a meaningful tool for studying and learning mathematics, then as Hanna (1995) suggested, efforts must be made "to enhance its role in the [undergraduate and secondary] classroom by finding more effective ways of using it as a vehicle to promote mathematical understanding" (p. 42).

The role of proof in promoting understanding was another aspect of proof that many teachers omitted as a characteristic of arguments that they found convincing. In light of the comments in the preceding paragraph, however, teachers' failure to include this aspect of proof may have been an artifact of the interview protocol

[4] For additional information on R. L. Moore and the method of teaching with which he is credited, see http://www.discovery.utexas.edu/rlm/index.html.

design. In particular, the arguments used during the interviews all provided support for statements that the teachers knew to be true (e.g., the sum of the angles in a triangle is 180°); as a result, there was not a genuine need for teachers to be convinced of the truth of a statement. Had there been such a need, teachers might have preferred those arguments that helped them to understand why a given statement is indeed true.

What Is Proof?

Many of the teachers studied did not seem to have robust beliefs about the meanings that they ascribed to the notion of proof. Every teacher talked about proof as an argument that demonstrates the truth of a statement and expressed the view that the demonstration of truth is a primary role of proof as well. Yet, a significant number of these same teachers seemed to believe that a proof is a fallible construct—that counterexamples or other contradictory evidence may exist—or they expressed some other measure of doubt about the generality of a proof. These teachers' views of proof were in stark contrast to views of proof as "not merely beyond reasonable doubt, but beyond *all* doubt" (Dunham, 1994, p. 117). Although Dunham's latter point may seem obvious, the infallibility (or fallibility) of a proof may not have been explicitly addressed or discussed during many teachers' undergraduate mathematics experiences. For example, not all schools require that their teacher education programs include the study of non-Euclidean geometries—a domain in which the issue naturally arises.

Many teachers also seemed to reach a stronger level of conviction regarding the truth of a proof's conclusion by testing it with empirical evidence. To some degree this need for confirmation is not surprising; Fischbein (1982) suggested that the need for additional confirmation primarily reflects differences in what is accepted as proof in everyday situations versus mathematical situations:

> The two basic ways of proving—the empirical and the logical—are not symmetrical, they do not have the same weight in our practical activity.... The concept of formal proof is completely outside the mainstream of behavior. A formal proof offers an absolute guarantee to a mathematical statement. Even a single practical check is superfluous. This way of thinking, knowing, and proving, basically contradicts the practical way of knowing which is permanently in search of additional confirmation. (p. 17)

Martin and Harel (1989) reached a similar conclusion in their study, suggesting that some individuals need to combine a deductive argument with empirical evidence to believe a particular conclusion. In Fischbein's study, as well as Martin and Harel's, however, the subjects had far less extensive mathematics backgrounds (middle grade students and prospective elementary teachers, respectively) than the subjects in the study reported here.

The teachers also displayed varying abilities in distinguishing between arguments that constituted proofs and those that did not; they tended to be very proficient at recognizing proofs but had more difficulty recognizing nonproofs. With regard to the latter, for example, several teachers accepted empirically based arguments as proofs. Further, many teachers accepted arguments as proofs seemingly on the basis of their judgments of the mechanics of an argument (e.g., correct symbolic manipulation) or of an argument's form (e.g., appears to be a proof by induction) rather than on the correctness of the reasoning presented.

The fact that these results are consistent with those found by other researchers (e.g., Harel & Sowder, 1998; Martin & Harel, 1989) suggests that such inadequate understandings of what constitutes proof may be widespread. It is also worth recalling that the arguments presented to teachers were specifically chosen so that the underlying mathematical concepts were not difficult; it was intended that the focus of the teachers would be on the argument presented rather than on trying to understand the mathematics used in producing the argument. One can imagine that making the content more difficult (i.e., undergraduate mathematics) might present additional obstacles to individuals in deciding what counts as proof. Such a change also raises an interesting question: How does the interaction of the mathematics content with the presented argument influence one's understanding of proof?

Finally, the teachers seemed to view the validity of arguments along a continuum, with the variability being a function of their sense of an argument's completeness. That is, the teachers tended to use one of two values (a rating of 3 or 4) rather than a single value (a rating of 4) for arguments that they identified as proofs. In some cases, their decision between a 3 and a 4 was dependent on the level of detail provided in the proof; in other cases, it was dependent on the level of knowledge required to understand the proof. It is interesting that such decisions reflect, to some extent, the fact that the discipline of mathematics does not have absolute criteria for what counts as proof—in the teachers' case, criteria for the degree of explicitness that is required and the mathematical results that are acceptable to use. It might have been interesting if teachers had been "forced" to rate an argument either as a proof or as a nonproof (i.e., using a 2-point rating scale). How would teachers have viewed the arguments that they originally rated as proofs but to which they assigned a rating of 3? Would teachers have weighed the arguments in a fashion similar to that in which mathematicians often weigh arguments offered as proof of a claim, asking, "Does the argument convincingly present the case that a formal proof exists and could be generated if so desired?"

What Is Convincing?

The characteristics of arguments that the teachers found to be most convincing seemed, in large part, to relate more to form than to substance. In other words, the majority of the characteristics that the teachers suggested concerned qualities related to an argument's form—features of the argument, the teacher's own familiarity with the argument, the amount of detail provided by the argument, or the particular method used in constructing the argument—rather than to the mathematical substance of the argument. In the area of mathematical substance, only two of the characteristics described by teachers—that an argument proves the general case and that an argument explains why a statement is true—spoke explicitly to the mathematics of an argument. Because the teachers were not specifically asked which arguments they found most *mathematically* convincing, they may have suggested arguments with characteristics resting on psychological grounds (i.e., characteristics that they found *personally* convincing). For example, several teachers found that they were most convinced by an empirically based argument, though they were fully aware that the argument was not a proof; it is unlikely, however, that these same teachers were *mathematically* convinced by such an argument. The difference between being convinced mathematically and being convinced personally or psychologically is underscored in the comments of Fischbein (1982) cited earlier and is an area that warrants further research. In addition, as noted earlier, had the statements being justified been unfamiliar to the teachers, they might have considered other characteristics of an argument (e.g., whether or not the argument promoted understanding).

CONCLUSION

The conclusion that teachers' conceptions of proof are somewhat limited should not be entirely surprising. After all, literature abounds that has documented limitations in teachers' conceptions in other mathematical domains (e.g., Ball, 1990; Even, 1993). In fact, such limitations have become a central concern for many teacher educators, as it is commonly acknowledged that teachers' subject matter conceptions are perhaps the most important influence on their instructional practice and ultimately on what their students learn (Borko & Putnam, 1996; Brophy, 1991). Thus, it seems clear that if teachers are to be successful in enhancing the role of proof in secondary school mathematics classrooms, then their conceptions of proof must be enhanced.

The responsibility for enhancing teachers' conceptions of proof lies with both mathematicians and mathematics educators, the parties who are chiefly responsible for the nature of teachers' experiences with proof and who, traditionally, have not adequately prepared teachers to succeed in enacting the lofty expectations set forth in reform documents (Ross, 1998). Of these two parties, university mathematics professors perhaps play the more significant role in shaping teachers' conceptions of proof. As Alibert and Thomas (1991) noted,

> [The] context in which students meet proofs in mathematics may greatly influence their perception of the value of proof. By establishing an environment in which students may see and experience first-hand what is necessary for them to convince others, of the truth or falsehood of propositions, proof becomes an instrument of personal value which they will be happier to use [*or teach*] in the future (p. 230).

Establishing such an environment in university mathematics classrooms may require that coursework "give conscious and perhaps overt attention to proof understanding, proof production, and proof appreciation as goals of instruction" (Harel & Sowder, 1998, p. 275). In short, teachers need, *as students*, to experience proof as a meaningful tool for studying and learning mathematics. Experiences of this nature may influence the conceptions of proof that they develop as teachers, and these ideas, in turn, may influence the experiences with proof their students will encounter in secondary school mathematics classrooms.

Although changing the nature of teachers' experiences with proof as students in mathematics courses may certainly enhance their conceptions of proof, such enhancement may be only a necessary but not a sufficient condition for enabling teachers to teach proof meaningfully to secondary school mathematics students. Future research needs to explore more fully the conceptions of proof that teachers must have as they help students learn to reason mathematically. What do teachers need to know about proof and how do they draw on and use this knowledge in the act of teaching? What conceptions of proof are necessary in selecting and designing tasks to present to students? Which are essential for making sense of and changing one's practice to more closely reflect reform recommendations about proof? As our understanding of the answers to the foregoing questions grows, we will be in a better position to support teachers in their efforts to make proof a more meaningful part of their classroom practices.

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